Non-Equilibrium States of a Photon Cavity Pumped by an Atomic Beam

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Abstract. We consider a beam of two-level randomly excited atoms that pass one-by-one through a one-mode cavity. We show that in the case of an ideal cavity, i.e. no leaking of photons from the cavity, the pumping by the beam leads to an unlimited increase in the photon number in the cavity. We derive an expression for the mean photon number for all times. Taking into account leaking of the cavity, we prove that the mean photon number in the cavity stabilizes in time. The limiting state of the cavity in this case exists and it is independent of the initial state. We calculate the characteristic functional of this non-quasi-free non-equilibrium state. We also calculate the total energy variation in both the ideal and the open cavities as well as the entropy production in the ideal cavity.

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Keywords. Non-equilibrium quantum theory, open quantum systems, repeated interactions, time dependent interactions, W^*-dynamical systems, energy-entropy variations, photon cavity.

Contents

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1. Model and Results

1.1. The Model

Our model consists of a beam $\mathcal{A}$ of two-level randomly excited atoms that pass one-by-one through a photon cavity $\mathcal{C}$. During the passage time τ the atom in the cavity interacts with the cavity field.

The cavity is a one-mode resonator described by a quantum harmonic oscillator with Hamiltonian $H_{\mathcal{C}} = \epsilon\, b^* b \otimes \mathbb{1}$ acting on the Hilbert space $\mathcal{H}_{\mathcal{C}}$. Here b^* and b stand for boson creation and annihilation operators with canonical commutation relations (CCR): $[b\,, b^*] = \mathbb{1}$, $[b\,, b] = [b^*, b^*] = 0$.

The beam of two-level atoms with energy levels 0 and $E > 0$, is described by a chain $H_{\mathcal{A}} = \sum_{k\geq 1} H_{\mathcal{A}_k}$ of individual atoms $\mathcal{A}_k$ with Hamiltonian $H_{\mathcal{A}_k} = \mathbb{1} \otimes E\, \eta_k$ in the Hilbert space $\mathcal{H}_{\mathcal{A}} = \otimes_{k\geq 1} \mathcal{H}_{\mathcal{A}_k}$. Here for any $k \geq 1$, $\mathcal{H}_{\mathcal{A}_k} = \mathbb{C}^2$ and the individual atomic operator $\eta_k := (\sigma^z + I)/2$, where σ^z is the third Pauli matrix. The eigenvectors $\psi_k^{\pm}$: $\eta_k \psi_k^+ = \psi_k^+$ and $\eta_k \psi_k^- = 0$, are interpreted as the excited and the ground states of the atom, respectively.

The initial state of the system is the product state of the cavity and the states of each individual atom:

$$\rho_{\mathcal{S}} := \rho_{\mathcal{C}} \otimes \bigotimes_{k\geq 1} \rho_k \ . \tag{1.1}$$

Here $\rho_{\mathcal{C}}$ is the initial state of the cavity, which we assume to be normal, i.e., given by a density matrix $\rho_{\mathcal{C}} \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$, the space of the trace-class operators on $\mathcal{H}_{\mathcal{C}}$, and $\rho_{\mathcal{A}} := \bigotimes_{k\geq 1} \rho_k$ is the state of the beam.

In this paper, we suppose that the atomic states $\{\rho_k\}_{k\geq 1}$ on the algebras $\{\mathfrak{C}_1(\mathcal{H}_{\mathcal{A}_k})\}_{k\geq 1}$ are diagonal and identical (*homogeneous* beam), hence of the form

$$\rho_k = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \ , \quad p := \mathrm{Tr}_{\mathcal{A}_k}(\eta_k\, \rho_k) \ . \tag{1.2}$$

The parameter $0 \leq p \leq 1$ denotes probability that atom in beam is in its excited state ψ_k^+.

For simplicity we consider our model in the regime when at any moment only *one* atom is present in the cavity. Physically, this corresponds to a special *tuning* of the cavity size l and the interatomic distance $d = l$. Then the cavity-atom interaction has piecewise constant time-dependence, and we take it of the form:

$$K_k(t) = \chi_{[(k-1)\tau, k\tau)}(t)\, \lambda\, (b^* + b) \otimes \eta_k \ . \tag{1.3}$$

Here $\chi_{\Delta}(x)$ is characteristic function of the set Δ.

Since the non-excited atom is not "visible" by the cavity ($K_k(t)(u \otimes \psi_k^-) = 0$ for any $u \in \mathrm{dom}(H_{\mathcal{C}})$), the *detuned* case $d > l$, when there is at most one atom inside the cavity at the same time, is also described by a piecewise constant interaction. This situation can be handled by a small modification of our arguments, but we will not consider this situation further in this paper.

Remark 1.1. It is convenient, albeit not compulsory, to stick with a quantum description of the atomic beam. If one restricts oneself to the atomic observables η_k, which generate a commutative algebra, one can also consider $H_{\mathcal{A}}$ and interaction (1.3) as matrix representation of the continuous-time Bernoulli process with time-unity τ. Then the its piecewise constant random realisations

$$\widehat{\eta}(t) = \sum_{k\geq 1} (\psi_k^{\pm}, \chi_{[(k-1)\tau,k\tau)}(t)\eta_k\psi_k^{\pm})_{\mathcal{H}_{\mathcal{A}_k}} \ , \tag{1.4}$$

are generated by random sequences of atomic operator $\eta_k := (\sigma^z + I)/2$ eigenvectors $\{\psi_k^{\pm}\}_{k\geq 1}$ with probabilities p and $1-p$ for eigenvalues 1 and 0 respectively. Here $(\cdot\,,\,\cdot)_{\mathcal{H}_{\mathcal{A}_k}}$ is the scalar product in the space $\mathcal{H}_{\mathcal{A}_k} = \mathbb{C}^2$.

Returning back to the quantum electrodynamic origin of the interaction (1.3) one observes that it is completely elastic, since the atomic system does not evolve. The atom remains in the same state throughout its interaction with the photon field. This may be interpreted as the limit of rigid atoms that "kick" the cavity mode, see [FJMa].

The Hamiltonian for the entire system $\mathcal{S}$ acts on the space $\mathcal{H}_{\mathcal{S}} := \mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{A}}$, and is given by the sum of the Hamiltonians of the cavity and the atoms, and the interaction between them:

$$\begin{aligned} H(t) &= H_{\mathcal{C}} + \sum_{k\geq 1}(H_{\mathcal{A}_k} + K_k(t)) \qquad (1.5)\\ &= \epsilon\, b^*b \otimes \mathbb{1} + \sum_{k\geq 1} \mathbb{1} \otimes E\,\eta_k + \sum_{k\geq 1} \chi_{[(k-1)\tau,k\tau)}(t)(\lambda\,(b^*+b)\otimes\eta_k)\,. \end{aligned}$$

Notice that for the time $t \in [(n-1)\tau, n\tau)$, $n \geq 1$, only the n-th atom interacts with the cavity and the system is *autonomous.*

Projected on the time invariant sequences of $\{\psi_k^{\pm}\}_{k\geq 1}$ the system (1.5) is a one-mode cavity Hamiltonian with random interaction driven by the 1-0 Bernoulli process (1.4).

1.2. Hamiltonian Dynamics of the Ideal Cavity

Let $t \in [(n-1)\tau, n\tau)$. Then the Hamiltonian (1.5) for the n-th atom in the cavity takes the form $H(t) = H_n$, where

$$H_n := \epsilon\, b^*b \otimes \mathbb{1} + \sum_{k\geq 1} \mathbb{1} \otimes E\,\eta_k + \lambda\,(b^*+b)\otimes\eta_n \quad . \tag{1.6}$$

Although the atomic beam is infinite, since we will assume that the initial state is normal and since up to any finite time t only a finite number of atoms have interacted with the cavity, we can describe the evolved system by *normal* states $\omega_{\mathcal{S}}(\cdot) := \mathrm{Tr}(\,\cdot\,\rho_{\mathcal{S}})$, which are defined by density matrices $\rho_{\mathcal{S}}$

from the space of the *trace-class* operators $\mathfrak{C}_1(\mathcal{H}_\mathcal{C} \otimes \mathcal{H}_\mathcal{A})$. Then the partial traces over $\mathcal{H}_\mathcal{A}$ and over $\mathcal{H}_\mathcal{C}$:

$$\begin{aligned} \omega_\mathcal{C}(\cdot) &:= \omega_\mathcal{S}(\cdot \otimes \mathbb{1}) = \mathrm{Tr}_\mathcal{C}(\,\cdot\, \mathrm{Tr}_\mathcal{A}\rho_\mathcal{S}) \ , \\ \omega_\mathcal{A}(\cdot) &:= \omega_\mathcal{S}(\mathbb{1} \otimes \cdot) = \mathrm{Tr}_\mathcal{A}(\,\cdot\, \mathrm{Tr}_\mathcal{C}\rho_\mathcal{S}) \ , \end{aligned} \tag{1.7}$$

define respectively the cavity and the beam states.

Since below we mostly deal with normal states, we will use the terms 'state' and 'density matrix' interchangeably, if this does not cause any confusion.

We suppose that initially our system is in a *product-state* $\rho_\mathcal{S}(t)\big|_{t=0} = \rho_\mathcal{C} \otimes \rho_\mathcal{A}$ of the sub-systems $\mathcal{C}$ and $\mathcal{A}$:

$$\omega_\mathcal{S}^0(\cdot) = \mathrm{Tr}(\,\cdot\, \rho_\mathcal{C} \otimes \rho_\mathcal{A}) \ \ . \tag{1.8}$$

For any states $\rho_\mathcal{C}$ on $\mathfrak{A}(\mathcal{H}_\mathcal{C})$ and $\rho_\mathcal{A}$ on $\mathfrak{A}(\mathcal{H}_\mathcal{A})$ the Hamiltonian dynamics of the system is defined by (1.5), or by the quantum time-dependent Liouvillian generator:

$$L(t)(\rho_\mathcal{S}(t)) := -i\,[H(t), \rho_\mathcal{S}(t)] \ . \tag{1.9}$$

Then the state $\rho_\mathcal{S}(t) := (\rho_\mathcal{C} \otimes \rho_\mathcal{A})(t)$ of the total system at the time t is a solution of the non-autonomous Cauchy problem corresponding to Liouville differential equation

$$\frac{d}{dt}\rho_\mathcal{S}(t) = L(t)(\rho_\mathcal{S}(t)) \ , \ \rho_\mathcal{S}(t)\big|_{t=0} = \rho_\mathcal{C} \otimes \rho_\mathcal{A} \ . \tag{1.10}$$

We denote by $\omega_\mathcal{S}^t(\cdot) := \mathrm{Tr}(\,\cdot\, \rho_\mathcal{S}(t))$ the system time evolution due to (1.10) for the initial product state $\omega_\mathcal{S}(\cdot)$ (1.8).

Notice that in general (see e.g. [BrRo1]) the Hamiltonian evolution (1.10) with time-dependent generator is a family of unitary mappings $\{U_{t,s}\}_{s\leq t}$ (called also evolution operators or propagators)

$$i\,\frac{d}{dt}U_{t,s} = H(t)\,U_{t,s} \ , \ s < t \ , \tag{1.11}$$

with the composition rule:

$$U_{t,s} = U_{t,r}\,U_{r,s} \ , \quad \text{for} \quad s \leq r \leq t \ . \tag{1.12}$$

Here $s, r, t \in \mathbb{R}$ and $U_{t,t} = \mathbb{1}$. Then solution of (1.10) is

$$\rho_\mathcal{S}(t) = U_{t,s}\,\rho_\mathcal{S}(s)\,U_{t,s}^{-1} =: T_{t,s}(\rho_\mathcal{S}(s)) \ , \tag{1.13}$$

where by (1.12) the mapping $T_{t,s}(\cdot) = T_{t,r}(T_{r,s}(\cdot))$.

For our model with the tuned repeated interactions the form of the evolution operator (1.12) and the solution (1.13) are considerably simplified. Indeed, our system (1.5) is *autonomous* for each interval $[(k-1)\tau, k\tau)$. Then by virtue of (1.6) and (1.9) the Liouvillian generator

$$L(t)(\cdot) = L_k(\cdot) = -i[H_k, \,\cdot\,] \ , \quad \text{for} \ \ t \in [(k-1)\tau, k\tau) \ , \ k \geq 1 \ , \tag{1.14}$$

is piecewise constant (time-independent). For any $t \geq 0$ one has the representation

$$t := n(t)\tau + \nu(t) \ , \ n(t) = [t/\tau] \ \ \text{and} \ \ \nu(t) \in [0, \tau) \ , \tag{1.15}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Then by (1.13) and (1.14) the solution of the Cauchy problem (1.10) for $t \in [(n-1)\tau, n\tau)$ takes the form:

$$\rho_{\mathcal{S}}(t) = T_{t,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}) := e^{\nu(t)L_n} e^{\tau L_{n-1}} \dots e^{\tau L_2} e^{\tau L_1}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}) \ . \tag{1.16}$$

By (1.12) and (1.14) the mapping $T_{t,0}$ is the composition

$$T_{t,0} = T_{t,(n-1)\tau} \prod_{k=n-1}^{k=1} T_k \tag{1.17}$$

of the *one-step* evolution maps defined as

$$T_k := T_{k\tau,(k-1)\tau} = e^{\tau L_k} \quad \text{and} \quad T_{t,(n-1)\tau} = e^{\nu(t)L_n} \ . \tag{1.18}$$

Consequently, the evolution of the initial state of the system (1.8) can be expressed as

$$\omega^t_{\mathcal{S}}(\cdot) = \mathrm{Tr}(\,\cdot\; T_{t,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})) \ . \tag{1.19}$$

The mathematical study of this kind of dynamics, for different types of repeated interaction $K_n(t)$ (1.5), was initiated in papers by Attal, Bruneau, Joye, Merkli, Pautrat, Pillet: [BJM1, APa, AJ1, AJ2, BJM2, BPi, BJM3]. These works provide a rigorous framework for such physical phenomenon as the "one-atom maser", see e.g. [MWM, FJMb]. The important mathematical aspect of repeated interactions is a piecewise constant (random) Hamiltonian dynamics like (1.16), which in certain limit of $\tau \to 0$ may produce an effective quantum Markovian dynamics, which drives the total system to an asymptotic state. In our case, we will be able to obtain the exact asymptotics of the dynamics of our simple model directly, i.e., without taking a limit.

In the next part of the present paper (Section 2) we exploit specific structure of the Hamiltonian dynamics (1.16) and a special form of interaction (1.3) to work out an effective Hamiltonian evolution of the perfectly isolated cavity $\mathcal{C}$. In this case, the pumping of the cavity by the atomic beam leads to an unlimited growth of the number of photons. This means that the limiting state of the cavity will not be described by a density matrix. For this case, our results concern evolution of the photon-number expectation $N(t)$ in the reduced by the partial trace (1.7), (1.19) time-dependent cavity state

$$\omega^t_{\mathcal{C}}(\cdot) := \omega^t_{\mathcal{S}}(\cdot \otimes \mathbb{1}) = \mathrm{Tr}_{\mathcal{C}}(\cdot\; \rho_{\mathcal{C}}(t)) \quad \text{for} \quad \rho_{\mathcal{C}}(t) := \mathrm{Tr}_{\mathcal{A}}\left(T_{t,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})\right) \ . \tag{1.20}$$

Note that in our model, see (1.5) and (1.6), the atomic states ρ_k do not evolve:

$$[\mathbb{1} \otimes \rho_k, H(t)] = 0 \ , \ k \geq 1 \ . \tag{1.21}$$

The form of the initial state (1.1) together with (1.16) and (1.20) determine a *discrete* time evolution for the cavity state: $\rho^{(n)}_{\mathcal{C}} := \rho_{\mathcal{C}}(t = n\tau)$, with the recursive formula

$$\begin{aligned}\rho_{\mathcal{C}}^{(n)} :=& \mathrm{Tr}_{\mathcal{A}}\rho_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{A}}[e^{\tau L_n}...e^{\tau L_2}e^{\tau L_1}(\rho_{\mathcal{C}} \otimes \bigotimes_{k=1}^{n} \rho_k)] \\ =& \mathrm{Tr}_{\mathcal{A}_n}[e^{\tau L_n}\{\mathrm{Tr}_{\mathcal{A}_{n-1}}...\mathrm{Tr}_{\mathcal{A}_1}e^{\tau L_{n-1}}...e^{\tau L_2}e^{\tau L_1}(\rho_{\mathcal{C}} \otimes \bigotimes_{k=1}^{n-1} \rho_k)\} \otimes \rho_n] \\ =& \mathrm{Tr}_{\mathcal{A}_n}[e^{\tau L_n}(\rho_{\mathcal{C}}^{(n-1)} \otimes \rho_n)] \ .\end{aligned} \tag{1.22}$$

Here we denoted the partial trace over the k-th atom space $\mathcal{H}_{\mathcal{A}_k}$ by $\mathrm{Tr}_{\mathcal{A}_k}(\cdot)$.

For any density matrix $\rho \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ corresponding to a normal state on the operator algebra $\mathfrak{A}(\mathcal{H}_{\mathcal{C}})$ we define the mapping $\mathcal{L} : \rho \mapsto \mathcal{L}(\rho)$, by

$$\mathcal{L}(\rho) := \mathrm{Tr}_{\mathcal{A}_k}(e^{\tau L_k}(\rho \otimes \rho_k)) = \mathrm{Tr}_{\mathcal{A}_k}[e^{-i\tau H_k}(\rho \otimes \rho_k)e^{i\tau H_k}] \ . \tag{1.23}$$

Here the last equality is due to (1.14). Note that the mapping (1.23) does not depend on $k \geq 1$, since the atomic states $\{\rho_k\}_{k\geq 1}$ are homogeneous (1.2). Then the cavity state at $t = k\tau$ is defined by the k-th power of $\mathcal{L}$:

$$\rho_{\mathcal{C}}^{(k)} = \mathcal{L}(\rho_{\mathcal{C}}^{(k-1)}) = \mathcal{L}^k(\rho_{\mathcal{C}}) \ . \tag{1.24}$$

Therefore, by (1.16), (1.22) and (1.24) one obtains that for any time $t = (n-1)\tau + \nu(t)$, where $\nu(t) \in [0, \tau)$, the cavity state is

$$\rho_{\mathcal{C}}(t) = \mathrm{Tr}_{\mathcal{A}_n}[e^{\nu(t)L_n}(\mathcal{L}^{n-1}(\rho_{\mathcal{C}}) \otimes \rho_n)] \ . \tag{1.25}$$

Our first result concerns the evolution of the expectation value $N(t)$ of the photon-number operator $\hat{N} := b^*b$ in the cavity at the time t:

$$N(t) := \omega_{\mathcal{C}}^t(b^*b) = \mathrm{Tr}_{\mathcal{C}}(b^*b \ \rho_{\mathcal{C}}(t)) \ . \tag{1.26}$$

For $t = n\tau$ the state of the cavity can be expressed using (1.24). Then (1.26) yields

$$N(n\tau) = \mathrm{Tr}_{\mathcal{C}}(b^*b \ \mathcal{L}^n(\rho_{\mathcal{C}})) \ . \tag{1.27}$$

In the theorem below we suppose that the initial cavity state $\omega_{\mathcal{C}}^t|_{t=0}(\cdot) = \omega_{\mathcal{C}}(\cdot)$ is gauge invariant, i.e. $e^{i\alpha\hat{N}}\rho_{\mathcal{C}}e^{-i\alpha\hat{N}} = \rho_{\mathcal{C}}$.

Theorem 1.2. *Let $\rho_{\mathcal{C}}$ be a gauge-invariant state. Then for a homogeneous beam the expectation value* (1.27) *of the photon-number operator in the cavity at the moment $t = n\tau$ is equal to*

$$N(t) = N(0) + n\,p(1-p)\,\frac{2\lambda^2}{\epsilon^2}\,(1 - \cos\epsilon\tau) + p^2\,\frac{2\lambda^2}{\epsilon^2}(1 - \cos n\epsilon\tau) \ . \tag{1.28}$$

If for the initial state $\rho_{\mathcal{C}}$ one takes in the theorem the Gibbs state for photons at the inverse temperature β:

$$\rho_{\mathcal{C}}^{\beta} = e^{-\beta\epsilon b^* b}/\mathrm{Tr}_{\mathcal{C}}e^{-\beta\epsilon b^* b} \ , \tag{1.29}$$

then the the first term in (1.28) is simply $N(0) = (e^{\beta\epsilon} - 1)^{-1}$.

If the initial cavity state $\rho_{\mathcal{C},r,\phi}$ is *not* gauge-invariant with the breaking gauge-invariance parameter: $r\,e^{i\phi} := \omega_{\mathcal{C},r,\phi}(b) = \mathrm{Tr}_{\mathcal{C}}(b\ \rho_{\mathcal{C},r,\phi})$, then instead of (1.28) one gets by Lemma 2.3

$$\omega^t_{\mathcal{C},r,\phi}(\hat{N}) = N(t) + p\,\frac{2\lambda r}{\epsilon}\left[\cos\phi - \cos(n\epsilon\tau - \phi)\right]\ ,\ t = n\tau\ , \tag{1.30}$$

where $\omega^t_{\mathcal{C},r,\phi}(\hat{N})\mid_{t=0} = N(0) \geq r^2$. The same lemma yields also the positivity of (1.30) for any time $t \geq 0$.

Remark 1.3. If all atoms in the beam are in the ground-state $\otimes_{k\geq 1}\psi^-_k$, one has: $(E\,\eta_k)\psi^-_k = 0$, and $\rho_k\mid_{p=0} = (\psi^-_k\,,\cdot)_{\mathcal{H}_{\mathcal{A}_k}}\psi^-_k$ is the projection operator. Then by (1.28) and (1.30) $\omega^t_{\mathcal{C},r,\phi}(\hat{N}) = \omega^t_{\mathcal{C},r,\phi}(\hat{N})\mid_{t=0}$. Since the mean cavity energy is $\varepsilon(t) = \mathrm{Tr}_{\mathcal{C}}(H_{\mathcal{C}}\ \rho_{\mathcal{C}}(t)) = \epsilon\,N(t)$. Therefore, there is no energy transfer from the beam to the cavity when $p = 0$. Whereas in the other extreme case, when all atoms are excited $\rho_k(\eta_k) = 1$, we obtain

$$\begin{aligned}\omega^{n\tau}_{\mathcal{C},r,\phi}(\hat{N}) =& \omega^{n\tau}_{\mathcal{C},r,\phi}|_{n=0}(\hat{N}) + \frac{2\lambda^2}{\epsilon^2}(1-\cos n\epsilon\tau) + \\ &+\frac{2\lambda r}{\epsilon}\left[\cos\phi - \cos(n\epsilon\tau - \phi)\right]\ .\end{aligned} \tag{1.31}$$

In case of the gauge-invariant initial state ($r = 0$) the mean value of the photon number (1.31) satisfies the estimate $N(t) \geq N(0)$ and it is oscillating between initial value $N(0)$ and $N(0) + 4\lambda^2/\epsilon^2$ with the cavity resonant frequency ϵ.

Remark 1.4. These *Rabi oscillations* (see e.g. [FJMb]) are a simple consequence of non-trivial Heisenberg time evolution of the cavity boson operators generated by Hamiltonian (1.6). For example, if $t \in [0,\tau)$, i.e. $n = 1$, then we obtain (cf. Lemma 4.1):

$$\begin{aligned} t : b\otimes \mathbb{1} \mapsto T^*_{t,0}(b\otimes\mathbb{1}) &:= e^{itH_1}(b\otimes\mathbb{1})e^{-itH_1} \\ &= e^{tL^*_{n=1}}(b\otimes\mathbb{1}) = e^{-i\epsilon t} b\otimes\mathbb{1} - \mathbb{1}\otimes\frac{\lambda}{\epsilon}\eta_1(1-e^{-i\epsilon t})\ . \end{aligned} \tag{1.32}$$

Here evolution maps on the algebra $\mathfrak{A}(\mathcal{H}_{\mathcal{C}})\otimes\mathfrak{A}(\mathcal{H}_{\mathcal{A}})$:

$$T^*_k = e^{\tau L^*_k}\ ,\ T^*_{t,(n-1)\tau} = e^{\nu(t)L^*_n}\ ,\ t = (n-1)\tau + \nu(t)\ , \tag{1.33}$$

are adjoint to (1.17), (1.18) by *duality* with respect to the state (1.8), see Appendix A.2. Here L^*_k denotes operator, which is adjoint to the Liouvillian generator (1.14). Then (1.19) and (1.17),(1.33) yield

$$\omega^t_{\mathcal{S}}(\cdot) = \mathrm{Tr}\,(T^*_{t,0}(\cdot)\ \rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}})\ \ ,\ \ T^*_{t,0} = \prod_{k=1}^{n-1} e^{\tau L^*_k}\ e^{\nu(t)L^*_n}\ . \tag{1.34}$$

Remark 1.5. The Hamiltonian evolution (1.32) *breaks down* the gauge invariance of the initial state $\omega_{\mathcal{C}}(\cdot) = \mathrm{Tr}_{\mathcal{C}}(\cdot\ \rho_{\mathcal{C}})$. Indeed, by (1.7) and (1.8) one gets for the initial gauge-invariant cavity state: $\omega_{\mathcal{C}}(b) = \omega_{\mathcal{S}}(b\otimes\mathbb{1}) = 0$. Since by (1.20) and (1.34) we have

$$\omega^t_{\mathcal{C}}(b) = \omega^t_{\mathcal{S}}(b\otimes\mathbb{1}) = \mathrm{Tr}\,(T^*_{t,0}(b\otimes\mathbb{1})\ \rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}})\ , \tag{1.35}$$

(1.2) and (1.32) imply that for $t \in [0, \tau)$

$$\omega_{\mathcal{C}}^{t}(b) = p\frac{\lambda}{\epsilon}(e^{-i\epsilon t} - 1) \ . \tag{1.36}$$

This property of dynamics is due to interaction (1.6), and it has also a non-trivial impact for the open cavity evolution.

1.3. Quantum Dynamics of Open Cavity

To make a contact of our model with a more physically realistic situation, we consider an open cavity. This allows the photons to leak *out* of the cavity, but also to diffuse *in* from the environment. In Theorems 1.6 and 1.8 we show that if the leaking rate is greater than the rate the environmental pumping, the photon number in the cavity stabilizes at a finite mean value.

We consider this case in the framework of Kossakowski-Lindblad extension of the Hamiltonian Dynamics to *irreversible* Quantum Dynamics (see Appendix A.1 and [AlFa, AJPII],) with time-dependent generator:

$$\begin{aligned} L_\sigma(t)(\rho_{\mathcal{S}}(t)) :=& - i[H(t), \rho_{\mathcal{S}}(t)] + \sigma_-\ b \otimes \mathbb{1}\ \rho_{\mathcal{S}}(t)\ b^* \otimes \mathbb{1} - \frac{\sigma_-}{2}\{b^* b \otimes \mathbb{1}, \rho_{\mathcal{S}}(t)\} \\ & + \sigma_+\ b^* \otimes \mathbb{1}\ \rho_{\mathcal{S}}(t)\ b \otimes \mathbb{1} - \frac{\sigma_+}{2}\{b\, b^* \otimes \mathbb{1}, \rho_{\mathcal{S}}(t)\} \ , \end{aligned} \tag{1.37}$$

for the *complete positive* evolution of total system, when the initial state is $\rho_{\mathcal{S}}(t = 0) = \rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}$ (1.8). For $\sigma_\mp > 0$ the σ-part of this generator (cf (1.9)) corresponds to the non-Hamiltonian part of dynamics. Here σ_- describes the rate of the photons *leaking* out of the open cavity into the environment, whereas σ_+ corresponds to the cavity *pumping* rate due to the photon infiltration from the environment. It has to be distinguished from the pumping mechanism due to the interaction with the atomic beam.

Note that similar to (1.10) the evolution of the state is defined by the solution of the non-autonomous Cauchy problem corresponding to the time-dependent generator (1.37). The proof of existence of this solution is a non-trivial problem (Appendix A.1), see for example [NVZ] for the case of lattice systems and bounded generators. For unbounded generators and for a general setting the proof that the Kossakowski-Lindblad generator in the form (1.37) corresponds to a properly defined continuous evolution is more involved [NZ, VWZ]. A separate problem is to prove that this map is *trace-preserving* and verifies the property of *complete positivity*, which are indispensable for correct description of the open system evolution, see Appendix A.1 and the references there.

To avoid these complications we consider the case of the *tuned* repeated interaction, when the Hamiltonian dynamics is piecewise autonomous for each interval $[(k-1)\tau, k\tau)$. Then for $t \in [(k-1)\tau, k\tau)$ the generator (1.37) has the form:

$$\begin{aligned} L_{\sigma,k}(\cdot) :=& - i[H_k, \cdot] + \sigma_-\ b \otimes \mathbb{1}\ (\cdot)\ b^* \otimes \mathbb{1} - \frac{\sigma_-}{2}\{b^* b \otimes \mathbb{1}, \cdot\} \\ & + \sigma_+\ b^* \otimes \mathbb{1}\ (\cdot)\ b \otimes \mathbb{1} - \frac{\sigma_+}{2}\{b\, b^* \otimes \mathbb{1}, \cdot\} \ , \quad k \geq 1 \ , \end{aligned} \tag{1.38}$$

and $L_{\sigma,k}\big|_{\sigma_{\mp}=0} = L_k$, see (1.14). Hence, the solution of the non-autonomous Cauchy problem

$$\frac{d}{dt}\rho_{\mathcal{S},\sigma}(t) = L_\sigma(t)(\rho_{\mathcal{S},\sigma}(t)) \ , \ \rho_{\mathcal{S},\sigma}(t)\big|_{t=0} = \rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}} \ , \tag{1.39}$$

for the piecewise *constant* generators (1.38), has the same form as for the ideal cavity, $\sigma_{\mp} = 0$ (1.16), (1.17):

$$\begin{aligned}\rho_{\mathcal{S},\sigma}(t) = T^{\sigma}_{t,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}) :=& \\ = e^{\nu(t)L_{\sigma,n}} e^{\tau L_{\sigma,n-1}} \ldots\, e^{\tau L_{\sigma,2}} e^{\tau L_{\sigma,1}}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}) \,.&\end{aligned} \tag{1.40}$$

Here $t = (n-1)\tau + \nu(t)$ and the mapping $T^{\sigma}_{t,0}$ is the composition of the one-step non-Hamiltonian evolution maps defined by (1.38):

$$T^{\sigma}_{k} := T^{\sigma}_{k\tau,(k-1)\tau} = e^{\tau L_{\sigma,k}} \quad \text{and} \quad T^{\sigma}_{t,(n-1)\tau} = e^{\nu(t)L_{\sigma,n}} \ . \tag{1.41}$$

By duality with respect to the initial state $\omega^0_{\mathcal{S}}(\cdot) = \mathrm{Tr}(\cdot\ \rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})$ one can now define the adjoint evolution mapping $\{(T^{\sigma}_{t,0})^*\}_{t\geq 0}$ by the relation

$$\omega^t_{\mathcal{S},\sigma}(A) = \mathrm{Tr}(A\ T^{\sigma}_{t,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})) = \omega^0_{\mathcal{S}}((T^{\sigma}_{t,0})^*(A)) \ , \tag{1.42}$$

for any $A \in \mathfrak{A}(\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{A}})$. Then similar to the nonleaky case (1.34) we obtain for any $t = (n-1)\tau + \nu(t)$ that

$$\omega^t_{\mathcal{S},\sigma}(\cdot) = \mathrm{Tr}\left((T^{\sigma}_{t,0})^*(\cdot)\ \rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}\right) \ , \ \ (T^{\sigma}_{t,0})^* = \prod_{k=1}^{n-1} e^{\tau L^*_{\sigma,k}}\ e^{\nu(t)L^*_{\sigma,n}} \ . \tag{1.43}$$

Here $\{L^*_{\sigma,k}\}_{k\geq 1}$ are generators, which are adjoint to (1.38).

Since the restriction of (1.40) to the dynamics of a cavity state $\rho \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ is the partial trace over beam states, the corresponding discrete evolution mappings (1.23) have to be modified for the open cavity as follows:

$$\mathcal{L}_\sigma(\rho) := \mathrm{Tr}_{\mathcal{A}_k}(e^{\tau L_{\sigma,k}}(\rho \otimes \rho_k)) \ . \tag{1.44}$$

The mapping (1.44) does not depend on $k \geq 1$, since the atomic states $\{\rho_k\}_{k\geq 1}$ are homogeneous (1.2). If $\rho_{\mathcal{C}} := \rho_{\mathcal{C},\sigma}(t)\ |_{t=0}$ is initial state of the open cavity, then similar to (1.25) we obtain by (1.44) for $\rho_{\mathcal{C},\sigma}(t)$ at the moment $t = (n-1)\tau + \nu(t)$:

$$\rho_{\mathcal{C},\sigma}(t) = \mathrm{Tr}_{\mathcal{A}_n}[e^{\nu(t)L_{\sigma,n}}(\mathcal{L}^{n-1}_\sigma(\rho_{\mathcal{C}}) \otimes \rho_n)] \ . \tag{1.45}$$

By (1.45) and (1.44) we obtain for the time-dependent open cavity state

$$\omega^t_{\mathcal{C},\sigma}(\cdot) := \omega^t_{\mathcal{S},\sigma}(\ \cdot \otimes \mathbb{1}) = \mathrm{Tr}_{\mathcal{C}}(\ \cdot\ \rho_{\mathcal{C},\sigma}(t)) \ , \tag{1.46}$$

where $\omega^t_{\mathcal{S},\sigma}(\cdot) := \mathrm{Tr}(\ \cdot\ T^{\sigma}_{t,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}))$ by (1.40). We also define

$$\omega^0_{\mathcal{C},\sigma}(\cdot) := \mathrm{Tr}_{\mathcal{C}}(\ \cdot\ \rho_{\mathcal{C}}) \quad \text{and} \quad \omega_{\mathcal{C},\sigma}(\cdot) := \lim_{t\to\infty} \omega^t_{\mathcal{C},\sigma}(\cdot) \ . \tag{1.47}$$

To study the infinite-time limit $\omega_{\mathcal{C},\sigma}(\cdot)$, we consider the functional

$$\omega_{\mathcal{C},\sigma}(W(\zeta)) = \lim_{t\to\infty} \omega^t_{\mathcal{C},\sigma}(W(\zeta)) \ , \tag{1.48}$$

generated by the Weyl operators on $\mathcal{H}_{\mathcal{C}}$:

$$W(\zeta) = e^{\frac{i}{\sqrt{2}}(\overline{\zeta}b + \zeta b^*)} \ , \ \zeta \in \mathbb{C} \ . \tag{1.49}$$

Notice that convergence (1.48) on the family of the Weyl operators guarantees the weak limit [BrRo1] of the states $\omega_{\mathcal{C}}^t(\cdot)$, when $t \to \infty$, see Appendices A.2 and A.3. The following theorem is our first result about the open cavity.

Theorem 1.6. *Let $\sigma_+ \geq 0$ and $\sigma_- - \sigma_+ > 0$. Then for any gauge-invariant initial cavity state $\rho_{\mathcal{C}}$ and for a homogenous atomic beam with parameter $p = \mathrm{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(\eta_n\ \rho_n)$, the limiting cavity state*

$$\omega_{\mathcal{C},\sigma}(\cdot) := \lim_{t\to\infty} \omega_{\mathcal{C},\sigma}^t(\cdot) \tag{1.50}$$

exists and it does not dependent on $\rho_{\mathcal{C}}$. Here the limit means trace-norm convergence of the sequence (1.45) *to a density matrix $\rho_{C,\sigma}$:*

$$\lim_{t\to\infty} \|\rho_{\mathcal{C},\sigma} - \rho_{\mathcal{C},\sigma}(t)\|_1 = 0, \tag{1.51}$$

where the norm $\|\cdot\|_1$ denotes the trace-norm on the space of trace-class operators $\mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$, see (5.31)*, Appendix A.2. The explicit form of the limiting functional* (1.48) *is*

$$\omega_{\mathcal{C},\sigma}(W(\zeta)) = e^{-\frac{|\zeta|^2}{4}\frac{\sigma_-+\sigma_+}{\sigma_--\sigma_+}} \times \tag{1.52}$$
$$\times \prod_{k=0}^{\infty} \left\{ p \exp \frac{1}{\sqrt{2}} \left(\frac{\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\overline{\zeta} - \frac{\lambda}{\overline{\mu}}(1-e^{-\overline{\mu}\tau})e^{-k\overline{\mu}\tau}\zeta \right) + 1 - p \right\},$$

where $\mu := i\epsilon + (\sigma_- - \sigma_+)/2$.

This result is obviously different from the case: $\sigma_- = \sigma_+ = 0$, when there is no regular limiting state, because the number of photons in the ideal cavity cavity unboundedly increases with time, see Theorem 1.2.

Corollary 1.7. Theorem 1.6 implies that $\omega_{C,\sigma}(\cdot)$ is a regular, normal (see Appendix A.5) and, in general, non-gauge-invariant state. One also sees that it is not quasi-free for $0 < p < 1$, but it obviously does for $p = 0$ or $p = 1$.

To verify these properties notice that by the Araki-Segal theorem (see Theorem 5.2), any *regular* state over $\mathrm{CCR}(\mathcal{H}_{\mathcal{C}})$ is uniquely defined by the characteristic functional $\{\zeta \mapsto \omega_{C,\sigma}(W(\zeta))\}_{\zeta\in\mathbb{C}}$. To check the conditions of this theorem we express the infinite product in (1.52) as a uniformly converging (by estimate (3.27)) infinite sum of terms that have a generic form:

$$e^{-\frac{|\zeta|^2}{4}\frac{\sigma_-+\sigma_+}{\sigma_--\sigma_+}}\ p^n(1-p)^m\ e^{ir_k(\zeta)}\ , \tag{1.53}$$

where

$$r_k(\zeta) := 2\ \mathrm{Im}\left([\lambda(1-e^{-\mu\tau})e^{-k\mu\tau}\overline{\zeta}]/\mu\sqrt{2} \right)\ . \tag{1.54}$$

By virtue of (5.55), apart from the normalization, expression (1.53) is nothing but the characteristic function of a *non-gauge-invariant* (see (1.54)) and *quasi-free* state on $\mathrm{CCR}(\mathcal{H}_{\mathcal{C}})$, which trivially verifies the Araki-Segal theorem. As a consequence, (1.52) defines a *normal* state since it is a convergent infinite convex combination of normal quasi-free states but, when $0 < p < 1$, the infinite combination of quasi-free states in (1.52) is *not* quasi-free.

Let $N_\sigma(t)$ be the expectation value of the photon-number operator b^*b in the open cavity at the time t as:

$$N_\sigma(t) := \omega^t_{\mathcal{C},\sigma}(b^*b) = \mathrm{Tr}_{\mathcal{C}}(b^*b\ \rho_{\mathcal{C},\sigma}(t))\ . \tag{1.55}$$

For the open cavity the result corresponding to the asymptotic behaviour of the photon number in the cavity takes the form.

Theorem 1.8. *Let $\sigma_- - \sigma_+ > 0$. Then for an arbitrary initial gauge-invariant cavity state $\rho_{\mathcal{C}}$ such that the initial mean-value of the photon number in the cavity is bounded :*

$$N_\sigma(0) = \omega^t_{\mathcal{C},\sigma}(b^*b)\ |_{t=0} = \mathrm{Tr}_{\mathcal{C}}(b^*b\ \rho_{\mathcal{C}}) < \infty\ , \tag{1.56}$$

we obtain that the expected number of photos at $t = (n-1)\tau + \nu(t)$ has the form:

$$\begin{aligned} N_\sigma(t) &= e^{-(\sigma_- - \sigma_+)t} N_\sigma(0) \\ &+ p\frac{\lambda^2}{|\mu|^2} e^{-(\sigma_- - \sigma_+)\tau}(1 - e^{\mu\tau})(1 - e^{\bar{\mu}\tau})\frac{1 - e^{-(\sigma_- - \sigma_+)t}}{1 - e^{-(\sigma_- - \sigma_+)\tau}} \\ &- p^2\frac{2\lambda^2}{|\mu|^2}\frac{1 - e^{-(\sigma_- - \sigma_+)t}}{1 - e^{-(\sigma_- - \sigma_+)\tau}}(1 - e^{-(\sigma_- - \sigma_+)\tau/2}\cos\epsilon\tau) \\ &+ p^2\frac{2\lambda^2}{|\mu|^2}(1 - e^{-(\sigma_- - \sigma_+)t/2}\cos\epsilon t) + \frac{\sigma_+}{\sigma_- - \sigma_+}(1 - e^{-(\sigma_- - \sigma_+)t}). \end{aligned} \tag{1.57}$$

Here $\mu := (\sigma_- - \sigma_+)/2 + i\,\epsilon$. The limit of the expected number of photons in the cavity is

$$\begin{aligned} \omega_{\mathcal{C},\sigma}(b^*b) &:= \lim_{t\to\infty} \omega^t_{\mathcal{C},\sigma}(b^*b) \\ &= p(1-p)\frac{2\lambda^2}{|\mu|^2}\frac{1 - e^{-(\sigma_- - \sigma_+)\tau/2}\cos\epsilon\tau}{1 - e^{-(\sigma_- - \sigma_+)\tau}} + p(2p-1)\frac{\lambda^2}{|\mu|^2} + \frac{\sigma_+}{\sigma_- - \sigma_+}\ . \end{aligned} \tag{1.58}$$

Remark 1.9. Note that the limit (1.58) satisfies the estimate from above:

$$\omega_{\mathcal{C},\sigma}(b^*b) \le \frac{2\lambda^2}{|\mu|^2}\frac{p(1-p)}{1 - e^{-(\sigma_- - \sigma_+)\tau/2}} + p(2p-1)\frac{\lambda^2}{|\mu|^2} + \frac{\sigma_+}{\sigma_- - \sigma_+}\ , \tag{1.59}$$

as well as the estimate from below:

$$\omega_{\mathcal{C},\sigma}(b^*b) \ge \frac{2\lambda^2}{|\mu|^2}\frac{p^2}{1 + e^{(\sigma_- - \sigma_+)\tau/2}} + \frac{\sigma_+}{\sigma_- - \sigma_+}\ , \tag{1.60}$$

There are several instructive cases that one can observe as corollaries of the above theorem:

- Let $p = 0$ (or $\lambda = 0$), i.e. no pumping by the atomic beam. Then the limiting value (1.58),(1.60) is $\sigma_+/(\sigma_- - \sigma_+)$, which coincides with the formula (5.42) in Appendix A.3. If in this case one put $\sigma_- > 0$ and consider $\sigma_+ \to 0$, then the limit (1.58) is equal to *zero*. This simply means that in the absence of atomic pumping the leaky cavity relaxes to the vacuum state empty of photons, or formally to the Gibbs state (5.44) with zero temperature $\beta_{\mathrm{cav}} = +\infty$.

- When $\lambda \neq 0$ and $p = 1$, still there is no (unlimited) pumping by the atomic beam but only the photon Rabi oscillations (1.28). Then for the open cavity $\sigma_- > \sigma_+ \geq 0$ the limit (1.58) and (1.59) give

$$\omega_{\mathcal{C},\sigma}(b^* b) = \frac{\lambda^2}{|\mu|^2} + \frac{\sigma_+}{\sigma_- - \sigma_+} \ . \tag{1.61}$$

  for any leaking $\sigma_- > 0$, including the *limit* of the ideal cavity when $\sigma_- \to 0$. Note that in the ideal cavity for the case $p = 1$ the mean-value of photons (1.28) is bounded, but oscillating. In other words, the limits: $t \to \infty$ and $\sigma_- \to 0$ do not commute.
- If $0 < p < 1$ and $\sigma_+ = 0$, then for the limit of ideal cavity (1.58) yields

$$\lim_{\sigma_- \to 0} \omega_{\mathcal{C},\sigma}(b^* b) = p(1-p)\, \frac{2\lambda^2}{\epsilon^2} \lim_{\sigma_- \to 0} \frac{(1 - \cos \epsilon\tau)}{1 - e^{-\tau\sigma_-}} \ . \tag{1.62}$$

  Hence, for the non-resonant case $\epsilon\tau \neq 2\pi s$, where $s \in \mathbb{Z}$, this limit is infinite, i.e. corresponding to the conclusion of the Theorem 1.2 about unlimited pumping of the ideal cavity. Indeed, (1.57) for $\sigma_+ = 0$ implies

$$\lim_{\sigma_- \to 0} \omega_{\mathcal{C},\sigma}^{t=n\tau}(b^* b)|_{\sigma_+ = 0} = N(n\tau) \ . \tag{1.63}$$

  Here $N(n\tau)$ coincides with (1.28), which diverges for the non-resonant case as $n\, p(1-p)\, 2\lambda^2 (1 - \cos \epsilon\tau)/\epsilon^2|_{n\to\infty}$, cf. (1.62).
- When $\sigma_+ > 0$ and $\sigma_- \to \sigma_+$, the limit (1.58) yields

$$\lim_{\sigma_- \to \sigma_+} \omega_{\mathcal{C},\sigma}(b^* b) = +\infty \ . \tag{1.64}$$

  It means that if the leaking and the environmental pumping have the same rate, the limiting state corresponds to the infinite temperature state, see (1.52) and (5.43), (5.44). On the Weyl operators this state is given by the Kronecker delta-functional

$$\omega_{\mathcal{C},\sigma}(W(\zeta)) = \begin{cases} 1 & \text{if } \zeta = 0, \\ 0 & \text{if } \zeta \neq 0. \end{cases}$$

  Since the Kronecker characteristic functional is not continuous, the corresponding state is not regular. Consequently, the Araki-Segal Theorem 5.2 is not applicable.
- Note that for $\sigma_+ > 0$ and $\sigma_- \to \sigma_+$ by (1.57) one gets for the expected number of photos at $t = n\tau$

$$\lim_{\sigma_- \to \sigma_+} \omega_{\mathcal{C},\sigma}^{n\tau}(b^* b) = N(n\tau) + n\,\tau\sigma_+ \ . \tag{1.65}$$

  Hence, in this case the pumping by the random non-resonant atomic beam (1.28) and by the environmental pumping due to $\sigma_+ > 0$, give the same *linear* rate for increasing of the mean number of photons in the cavity. Consequently, for $t = n\tau \to \infty$ the infinite photon number cavity state coincides with the infinite-temperature state that we discussed above, cf. (5.44).

This concludes the description of our main results. The rest of the paper is organized as follows. In Section 2 we use the specific properties of our model to diagonalise it, which is the key for the further analysis. Then we give the proof of Theorem 1.2 for the Hamiltonian dynamics of the ideal cavity.

In Section 3 we present our results for the case of the leaking cavity described by the Kossakowski-Lindblad irreversible quantum dynamics, i.e., Theorems 1.6 and 1.8.

The results concerning Energy-Entropy relations are presented in Section 4. There we calculate the energy variation (Theorem 4.2 and Theorem 4.8) and obtain a formula for the entropy production (formula (4.47)) for the ideal cavity.

The Section 5 is reserved for comments, remarks and open problems.

For the reader convenience we collect in an Appendix certain results and definitions necessary for the main text.

2. Hamiltonian Dynamics: The Ideal Cavity

Here we consider the case of the ideal cavity, i.e. $\sigma_- = \sigma_+ = 0$. In this case, the discrete evolution map $\mathcal{L}_{\sigma=0} = \mathcal{L}$ is given by (1.23). Recall that $p = \mathrm{Tr}(\eta_k \rho_k)$ for the homogeneous atomic states $\{\rho_k\}_{k\geq 1}$ in the beam.

Our first result concerns the expectation of the photon-number operator $\hat{N} = b^* b$ in the cavity (1.26). For $t = n\tau$ this expectation involves the calculation of $\mathcal{L}^n(\rho)$ (1.27). Instead, we use the n-th power of the *adjoint* operator $\mathcal{L}^*$ defined by duality with respect to the cavity state $\omega_{\mathcal{C}}(\cdot) = \mathrm{Tr}_{\mathcal{C}}(\cdot\ \rho_{\mathcal{C}})$, see (1.27) and Remark 2.2.

Let $\widehat{S}_k$ be $*$-isomorphism (unitary shift) on the algebra $\mathfrak{A}(\mathcal{H}_{\mathcal{C}}) \otimes \mathfrak{A}(\mathcal{H}_{\mathcal{A}_k})$ defined by

$$\widehat{S}_k(\cdot) := e^{iV_k}(\cdot)\, e^{-iV_k}\ ,\ V_k := \lambda(b^* - b) \otimes \eta_k / i\epsilon\ . \tag{2.1}$$

Since $\eta_k^2 = \eta_k$ and $\widehat{S}_k(\mathbb{1} \otimes \eta_k) = \mathbb{1} \otimes \eta_k$, whereas

$$\widehat{S}_k(b \otimes \mathbb{1}) = b \otimes \mathbb{1} - \mathbb{1} \otimes \frac{\lambda}{\epsilon}\eta_k\ ,\ \widehat{S}_k(b^* \otimes \mathbb{1}) = b^* \otimes \mathbb{1} - \mathbb{1} \otimes \frac{\lambda}{\epsilon}\eta_k\ , \tag{2.2}$$

the transformation (2.1) of the Hamiltonian (1.6) gives

$$\widehat{H}_k := \widehat{S}_k(H_k) = \epsilon\ b^* b \otimes \mathbb{1} + \mathbb{1} \otimes (E - \frac{\lambda^2}{\epsilon})\, \eta_k + \sum_{s\geq 1 : s \neq k} \mathbb{1} \otimes E\, \eta_s\ . \tag{2.3}$$

Notice that the dynamics generated by $\widehat{H}_k$ (or by H_k) leaves the atomic operator $\eta_k = (\sigma^z + I)/2$ invariant

$$e^{i\tau \widehat{H}_k}(\mathbb{1} \otimes \eta_k) e^{-i\tau \widehat{H}_k} = \mathbb{1} \otimes \eta_k\ .$$

Similarly to the unitary shift (2.1), we define on $\mathfrak{A}(\mathcal{H}_{\mathcal{C}})$ the $*$-isomorphism:

$$S(\cdot) := e^{iV}(\cdot)\, e^{-iV}\quad ,\quad V := \lambda(b^* - b)/i\epsilon\ . \tag{2.4}$$

Lemma 2.1. *For any state ρ on $\mathfrak{A}(\mathcal{H}_{\mathcal{C}})$ the one-step evolution mapping $\mathcal{L}$* (1.23) *has the form:*

$$\mathcal{L}(\rho) = p\; S^{-1}(e^{-i\tau\epsilon b^* b} S(\rho) e^{i\tau\epsilon b^* b}) + (1-p)\; e^{-i\tau\epsilon b^* b} \rho\, e^{i\tau\epsilon b^* b}\;. \tag{2.5}$$

Here $p = \mathrm{Tr}_{\mathcal{H}_{\mathcal{A}_k}}(\eta_k\;\rho_k)$ is the probability to find the k-th atom in the excited state with energy E.

Proof. Using the shift transformation (2.1) and the cyclicity of trace, one can re-write (1.23) as

$$\mathcal{L}(\rho) = \mathrm{Tr}_{\mathcal{A}_k}[\widehat{S}_k^{-1}(e^{-i\tau \widehat{H}_k} \widehat{S}_k(\rho\otimes\rho_k)\, e^{i\tau\widehat{H}_k})]\;. \tag{2.6}$$

Since the operator $\eta_k = (\sigma^z + I)/2$ is idempotent ($\eta_k^2 = \eta_k$) and since it commutes with ρ_k (1.2), the combination of expansions of (2.1), (2.4) with definitions of V_k and V gives

$$\widehat{S}_k(\rho\otimes\rho_k) = S(\rho)\otimes\eta_k\rho_k + \rho\otimes(I-\eta_k)\rho_k\;, \tag{2.7}$$

$$\widehat{S}_k^{-1}(\rho\otimes\rho_k) = S^{-1}(\rho)\otimes\eta_k\rho_k + \rho\otimes(I-\eta_k)\rho_k\;. \tag{2.8}$$

Therefore, plugging (2.7) into (2.6) we obtain

$$\begin{aligned}\mathcal{L}(\rho) =& \mathrm{Tr}_{\mathcal{A}_k}[\widehat{S}_k^{-1}(e^{-i\tau\widehat{H}_k}(S(\rho)\otimes\eta_k\rho_k)e^{i\tau\widehat{H}_k})] \\ &+ \mathrm{Tr}_{\mathcal{A}_k}[\widehat{S}_k^{-1}(e^{-i\tau\widehat{H}_k}(\rho\otimes(I-\eta_k)\rho_k)e^{i\tau\widehat{H}_k})]\;.\end{aligned}$$

Now, the diagonal form (2.3) of the Hamiltonian $\widehat{H}_k$ and (2.8) for $\widehat{S}_k^{-1}$, imply:

$$\begin{aligned}\mathcal{L}(\rho) =& \mathrm{Tr}_{\mathcal{A}_k}[S^{-1}(e^{-i\tau\epsilon b^*b}S(\rho)e^{i\tau\epsilon b^*b})\otimes\eta_k\rho_k] \\ &+ \mathrm{Tr}_{\mathcal{A}_k}[e^{-i\tau\epsilon b^*b}S(\rho)e^{i\tau\epsilon b^*b}\otimes(I-\eta_k)\eta_k\rho_k] \\ &+ \mathrm{Tr}_{\mathcal{A}_k}[S^{-1}(e^{-i\tau\epsilon b^*b}\rho\, e^{i\tau\epsilon b^*b})\otimes\eta_k(I-\eta_k)\rho_k] \\ &+ \mathrm{Tr}_{\mathcal{A}_k}[e^{-i\tau\epsilon b^*b}\rho e^{i\tau\epsilon b^*b}\otimes(I-\eta_k)\rho_k]\;.\end{aligned}$$

Since $(I-\eta_k)\eta_k = 0$ and $p = \mathrm{Tr}_{\mathcal{A}_k}[\mathbb{1}\otimes\eta_k\rho_k]$, one gets (2.5). □

Remark 2.2. (a) To calculate the expectation of the photon-number operator $\hat{N} = b^*b$ at $t = n\tau$ using (1.27), we would need to find the action of the n-th power $\mathcal{L}^n(\rho)$ of the operator (2.6).
We show that in fact it is easier to calculate this mean value using the n-th power of the adjoint (or *dual*) mapping $\mathcal{L}^*$. For any bounded operator $A \in \mathcal{B}(\mathcal{H}_{\mathcal{C}})$ and $\rho\in\mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$, it is defined by relation

$$\mathrm{Tr}_{\mathcal{C}}(\mathcal{L}^*(A)\rho) := \mathrm{Tr}_{\mathcal{C}}(A\;\mathcal{L}(\rho))\;, \tag{2.9}$$

see Appendix A.2 for discussion and details.
(b) Using invariance of the atomics states (1.21) one can obtain explicit expression for the one-step dual mapping $\mathcal{L}^*$. Indeed, by (1.21) and by (1.23)

$$\begin{aligned}\mathrm{Tr}_{\mathcal{C}}(A\;\mathcal{L}(\rho)) =& \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{A}_k}}\{(A\otimes\mathbb{1})\; e^{-i\tau H_k}(\rho\otimes\mathbb{1})(\mathbb{1}\otimes\rho_k)e^{i\tau H_k}\} \qquad (2.10)\\ =& \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{A}_k}}\{(\mathbb{1}\otimes\rho_k)e^{i\tau H_k}(A\otimes\mathbb{1})e^{-i\tau H_k}(\rho\otimes\mathbb{1})\} \\ =& \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{A}_k}}\{e^{i\tau H_k}(A\otimes\rho_k)\; e^{-i\tau H_k}(\rho\otimes\mathbb{1})\}\;,\end{aligned}$$

where we used cyclicity of the full trace $\mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{A}_k}}$. Hence, (2.9) together with (2.10) yield expression for the one-step mapping

$$\mathcal{L}^*(A) = \mathrm{Tr}_{\mathcal{A}_k}\{e^{i\tau H_k}(A\otimes\rho_k)e^{-i\tau H_k}\}\ , \tag{2.11}$$

which according to (1.23) is independent of $k \geq 1$.
(c) Let $\rho \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ be density matrix such that $\mathrm{Tr}_{\mathcal{C}}(\mathcal{P}(b,b^*)\,\mathcal{L}(\rho)) < \infty$ for any polynomial $\mathcal{P}(b,b^*)$. Then one can extend definition (2.9) of $\mathcal{L}^*$ to the class of unbounded observables, which contains $\mathcal{P}(b,b^*)$ and in particular the photon-number operator b^*b. The advantage to use the adjoint mapping $\mathcal{L}^*$ is that its consecutive applications do not increase the degree of polynomials in variables b and b^*.

Now, applying to (2.11) the same line of reasoning as in the proof of Lemma 2.1, we find explicit expression for the adjoint one-step mapping

$$\mathcal{L}^*(A) = \ p\,S^{-1}(e^{i\tau\epsilon b^*b}S(A)e^{-i\tau\epsilon b^*b}) + \ (1-p)e^{i\tau\epsilon b^*b}Ae^{-i\tau\epsilon b^*b}\ . \tag{2.12}$$

Note that alternatively one can obtain (2.12) using (2.5) and definition (2.9).

Lemma 2.3. *For $A = b^*b$ and for the adjoint operator $\mathcal{L}^*$ defined by* (2.12) *we obtain:*

$$\begin{aligned}(\mathcal{L}^*)^n(b^*b) &= b^*b + p\ \frac{\lambda}{\epsilon}[(1-e^{ni\epsilon\tau})\ b^* + (1-e^{-ni\epsilon\tau})\ b] \\ &\quad + n\,p(1-p)\ \frac{2\lambda^2}{\epsilon^2}(1-\cos\epsilon\tau) + p^2\ \frac{2\lambda^2}{\epsilon^2}(1-\cos n\epsilon\tau)\ .\end{aligned} \tag{2.13}$$

Proof. Since $*$-isomorphism (2.4) is a shift transformation, one gets

$$S(b^*b) = (b^* - \lambda/\epsilon)(b-\lambda/\epsilon)\ .$$

The CCR relations for b^* and b yield:

$$e^{i\tau\epsilon b^*b}b^*e^{-i\tau\epsilon b^*b} = e^{i\tau\epsilon}b^*\ ,\ e^{i\tau\epsilon b^*b}be^{-i\tau\epsilon b^*b} = e^{-i\tau\epsilon}b\ .$$

Consequently,

$$\begin{aligned}&S^{-1}(e^{i\tau\epsilon b^*b}S(b^*b)e^{-i\tau\epsilon b^*b}) = S^{-1}((e^{i\tau\epsilon}b^* - \lambda/\epsilon)(e^{-i\tau\epsilon}b-\lambda/\epsilon)) \\ &= (e^{i\tau\epsilon}b^* - (1-e^{i\tau\epsilon})\lambda/\epsilon)(e^{-i\tau\epsilon}b - (1-e^{-i\tau\epsilon})\lambda/\epsilon).\end{aligned} \tag{2.14}$$

Hence, (2.14) implies for (2.12)

$$\begin{aligned}\mathcal{L}^*(b^*b) &= p\,(e^{i\epsilon\tau}b^* - (1-e^{i\epsilon\tau})\lambda/\epsilon)(e^{-i\epsilon\tau}b - (1-e^{-i\epsilon\tau})\lambda/\epsilon) \\ &\quad + (1-p)\,b^*b \\ &= b^*b + p\frac{\lambda}{\epsilon}(1-e^{i\epsilon\tau})b^* + p\frac{\lambda}{\epsilon}(1-e^{-i\epsilon\tau})b + p\frac{2\lambda^2}{\epsilon^2}(1-\cos\epsilon\tau)\ ,\end{aligned} \tag{2.15}$$

which coincides with expression (2.13) for $n = 1$. If we insert in (2.12) $A = b^*$, and then $A = b$, we obtain correspondingly:

$$\mathcal{L}^*(b^*) = e^{i\epsilon\tau}b^* - p(1-e^{i\epsilon\tau})\lambda/\epsilon\ , \tag{2.16}$$

$$\mathcal{L}^*(b) = e^{-i\epsilon\tau}b - p(1-e^{-i\epsilon\tau})\lambda/\epsilon\ . \tag{2.17}$$

Now one can use induction. Since $(\mathcal{L}^*)^n(b^*b) = \mathcal{L}^*((\mathcal{L}^*)^{n-1}(b^*b))$, we can apply (2.15)-(2.17) to (2.13) for $n-1$ to check the formula (2.13) for the n-th power. □

Remark 2.4. As we indicated in Remark 1.5 the evolution $\mathcal{L}$ *breaks* the gauge invariance of the initial cavity state $\rho_{\mathcal{C}}$. For $n=1$ the gauge breaking parameters $r_{n=1}$ and $\phi_{n=1}$ are defined by (1.36), or by (2.9), (2.17):

$$\omega_{\mathcal{C}}^{\tau}(b) = \mathrm{Tr}_{\mathcal{C}}(\mathcal{L}^*(b)\rho_{\mathcal{C}}) = p\frac{\lambda}{\epsilon}(e^{-i\epsilon\tau}-1) = r_1\, e^{i\,\phi_1} \ . \tag{2.18}$$

Formulae (2.16),(2.17) allow to calculate r_n and ϕ_n for any $n \geq 1$. Iterating them one obtains

$$(\mathcal{L}^*)^n(b^*) = e^{in\epsilon\tau}b^* - p(1-e^{in\epsilon\tau})\lambda/\epsilon \ , \tag{2.19}$$

$$(\mathcal{L}^*)^n(b) = e^{-in\epsilon\tau}b - p(1-e^{-in\epsilon\tau})\lambda/\epsilon \ . \tag{2.20}$$

Therefore, the gauge breaking parameters r_n and ϕ_n are defined by equation:

$$\omega_{\mathcal{C}}^{n\tau}(b) = \mathrm{Tr}_{\mathcal{C}}((\mathcal{L}^*)^n(b)\rho_{\mathcal{C}}) = p\frac{\lambda}{\epsilon}(e^{-in\epsilon\tau}-1) =: r_n\, e^{i\,\phi_n} \ . \tag{2.21}$$

Proof. (of Theorem 1.2) To find the mean-value of the number of photons in the cavity at the time $t = n\tau$, we calculate the expectation of (2.13) in the initial cavity state $\omega_{\mathcal{C}}(\cdot) = \mathrm{Tr}_{\mathcal{C}}(\cdot\,\rho_{\mathcal{C}})$, (2.9). Since $\rho_{\mathcal{C}}$ is supposed to be *gauge-invariant*, this yields

$$\begin{aligned} N(t) =& \mathrm{Tr}_{\mathcal{C}}(b^*b\ \mathcal{L}^n(\rho_{\mathcal{C}})) = \mathrm{Tr}_{\mathcal{C}}((\mathcal{L}^*)^n(b^*b)\,\rho_{\mathcal{C}}) \qquad (2.22)\\ =& N(0) + n\,p(1-p)\ \frac{2\lambda^2}{\epsilon^2}\ (1-\cos\epsilon\tau) + p^2\frac{2\lambda^2}{\epsilon^2}(1-\cos n\epsilon\tau) \ , \end{aligned}$$

which coincides with (1.28) for $t = n\tau$. □

Corollary 2.5. If the initial cavity state $\rho_{\mathcal{C},r,\phi}$ is not gauge-invariant, then (1.30) and (2.22) give for $t = n\tau$

$$\begin{aligned} N(t) =& N(0) + n\,p(1-p)\ \frac{2\lambda^2}{\epsilon^2}\ (1-\cos\epsilon\tau) + p^2\frac{2\lambda^2}{\epsilon^2}(1-\cos n\epsilon\tau) \\ &+p\,\frac{2\lambda r}{\epsilon}\,[\cos\phi - \cos(n\epsilon\tau-\phi)] \ . \end{aligned} \tag{2.23}$$

Remark 2.6. Theorem 1.2 implies that the pumping effect is non-trivial (i.e. $N(t)$ is increasing to infinity) only for the beam of randomly exited atoms, i.e. $0 < p < 1$. There is no *unlimited* pumping of cavity by the non-exited, $p = 0$, or totaly exited, $p = 1$, atomic beams. Note that by (1.5) the mean value of the photon cavity energy is defined by $\mathcal{E}_{\mathcal{C}}(t) := \epsilon N(t)$. Hence, its unlimited increasing is due to the kicking by randomly excited chain of rigid atoms, which are *pushed* through the cavity.

3. Irreversible Quantum Dynamics: The Open Cavity

In a real physical cavity it is always possible for photons to leak *out* of the cavity as well as to diffuse *in* from the environment. This realistic situation is modeled by what is known as an *open* cavity. In the present section we include these effects, via the Kossakowski-Lindblad extension (1.37) of the Hamiltonian dynamics (1.5) with generator (1.9). Below we consider the case of the open cavity with *dominating* photon leaking: $\sigma_- > \sigma_+ \geq 0$, although our results allow also to study certain limiting cases, see Remark 1.9.

Similar to the ideal cavity, see (2.5) in Lemma 2.1, we first find a new expression for the one-step evolution mapping in the case of the open cavity (1.44).

Lemma 3.1. *For any state $\rho \in \mathfrak{C}_1(\mathcal{H}_C)$ the one-step evolution mapping* (1.44) *for the open cavity has the form:*

$$\mathcal{L}_\sigma(\rho) = p\, S^{-1}(e^{\tau L_{\lambda,\sigma}}(S(\rho))) + (1-p)\, e^{\tau L_{0,\sigma}}(\rho). \tag{3.1}$$

Here S is defined by (2.4) *and $L_{\lambda,\sigma}$ acts on $\mathfrak{A}(\mathcal{H}_C)$ as follows*

$$\begin{aligned} L_{\lambda,\sigma}(\rho) := &- i[\epsilon b^* b, \rho] \\ &+ \sigma_-(b-\lambda/\epsilon)\rho(b^*-\lambda/\epsilon) - \frac{\sigma_-}{2}\{(b^*-\lambda/\epsilon)(b-\lambda/\epsilon), \rho\} \\ &+ \sigma_+(b^*-\lambda/\epsilon)\rho(b-\lambda/\epsilon) - \frac{\sigma_+}{2}\{(b-\lambda/\epsilon)(b^*-\lambda/\epsilon), \rho\}, \end{aligned} \tag{3.2}$$

with $L_{0,\sigma} := L_{\lambda=0,\sigma}$.

Proof. Using $*$-isomorphisms (2.1),(2.4) and equations (2.7),(2.8) we define instead of generator (1.38) the following operator:

$$\begin{aligned} &\widehat{L}_{\sigma,k}(\rho\otimes\rho_k) := \widehat{S}_k(L_{\sigma,k}(\widehat{S}_k^{-1}(\rho\otimes\rho_k))) \\ &= -i[\epsilon b^* b \otimes \mathbb{1} + \mathbb{1}\otimes(E-\lambda^2/\epsilon)\eta_k, \rho\otimes(\eta_k\rho_k + (I-\eta_k)\rho_k)] \\ &+ \Big(\sigma_-(b-\lambda/\epsilon)\rho(b^*-\lambda/\epsilon) - \frac{\sigma_-}{2}\{(b^*-\lambda/\epsilon)(b-\lambda/\epsilon),\rho\}\Big)\otimes\eta_k\rho_k \\ &+ \Big(\sigma_+(b^*-\lambda/\epsilon)\rho(b-\lambda/\epsilon) - \frac{\sigma_+}{2}\{(b-\lambda/\epsilon)(b^*-\lambda/\epsilon),\rho\}\Big)\otimes\eta_k\rho_k \\ &+ \Big(\sigma_- b\rho b^* - \frac{\sigma_-}{2}\{bb^*,\rho\}\Big)\otimes(I-\eta_k)\rho_k + \Big(\sigma_+ b^*\rho b - \frac{\sigma_-}{2}\{b^* b,\rho\}\Big)\otimes(I-\eta_k)\rho_k \\ &= L_{\lambda,\sigma}(\rho)\otimes\eta_k\rho_k + L_{0,\sigma}(\rho)\otimes(I-\eta_k)\rho_k\ . \end{aligned} \tag{3.3}$$

With help of (3.3) and (2.7),(2.8) we obtain the representation

$$\begin{aligned} &e^{\tau L_{\sigma,k}}(\rho\otimes\rho_k) = \widehat{S}_k^{-1}(e^{\tau\widehat{L}_{\sigma,k}}(\widehat{S}_k(\rho\otimes\rho_k))) \\ &= \widehat{S}_k^{-1}(e^{\tau\widehat{L}_{\sigma,k}}(S(\rho)\otimes\eta_k\rho_k + \rho\otimes(I-\eta_k)\rho_k)) \\ &= \widehat{S}_k^{-1}(e^{\tau L_{\lambda,\sigma}}(S(\rho)\otimes\eta_k\rho_k)) + \widehat{S}_k^{-1}(e^{\tau L_{0,\sigma}}(\rho)\otimes(I-\eta_k)\rho_k) \\ &= S^{-1}(e^{\tau L_{\lambda,\sigma}}(S(\rho)))\otimes\eta_k\rho_k + e^{\tau L_{0,\sigma}}(\rho)\otimes(I-\eta_k)\rho_k\ . \end{aligned} \tag{3.4}$$

Therefore, the one-step mapping (1.44) takes the form

$$\begin{aligned}\mathcal{L}_\sigma(\rho) &= \mathrm{Tr}_{\mathcal{A}_k}(e^{\tau L_{\sigma,k}}(\rho\otimes\rho_k)) \\ &= p\, S^{-1}(e^{\tau L_{\lambda,\sigma}}(S(\rho))) + (1-p)\, e^{\tau L_{0,\sigma}}(\rho)\ ,\end{aligned}$$

which is claimed by Lemma in (3.1). □

Corollary 3.2. To define the adjoint mapping $\mathcal{L}^*_\sigma$ we use duality relation (2.9) for (3.1) and any bounded operator $A \in \mathcal{B}(\mathcal{H}_\mathcal{C})$:

$$\mathrm{Tr}_\mathcal{C}(\mathcal{L}^*_\sigma(A)\rho) := \mathrm{Tr}_\mathcal{C}(A\ \mathcal{L}_\sigma(\rho))\ . \tag{3.5}$$

Then by definitions (2.4), (3.5) and by explicit form (3.1) of the operator $\mathcal{L}_\sigma(\rho)$ one gets that

$$\mathcal{L}^*_\sigma(A) = p\, S^{-1}((e^{\tau L_{\lambda,\sigma}})^*(S(A))) + (1-p)\,(e^{\tau L_{0,\sigma}})^*(A)\ . \tag{3.6}$$

If for adjoint operators $(e^{\tau L_{\lambda,\sigma}})^*$ and $(e^{\tau L_{0,\sigma}})^*$ we define the corresponding adjoint generators by

$$e^{\tau L^*_{\lambda,\sigma}} := (e^{\tau L_{\lambda,\sigma}})^* \ \text{ and }\ e^{\tau L^*_{0,\sigma}} := (e^{\tau L_{0,\sigma}})^*\ , \tag{3.7}$$

then (3.1),(3.2) and (3.5)-(3.7) allow to find them explicitly:

$$\begin{aligned}L^*_{\lambda,\sigma}(A) &= i[\epsilon b^*b, A] + \frac{\sigma_-}{2}(b^*-\lambda/\epsilon)[A,b] + \frac{\sigma_-}{2}[b^*,A](b-\lambda/\epsilon) \\ &\quad + \frac{\sigma_+}{2}(b-\lambda/\epsilon)[A,b^*] + \frac{\sigma_+}{2}[b,A](b^*-\lambda/\epsilon)\ ,\end{aligned} \tag{3.8}$$

$$\begin{aligned}L^*_{0,\sigma}(A) &= i[\epsilon b^*b, A] + \frac{\sigma_-}{2}b^*[A,b] + \frac{\sigma_-}{2}[b^*,A]b \\ &\quad + \frac{\sigma_+}{2}b[A,b^*] + \frac{\sigma_+}{2}[b,A]b^*\ ,\end{aligned} \tag{3.9}$$

by straightforward calculations.

Remark 3.3. As we indicated in Remark 2.2 (c), one can extend the adjoint one-step mapping $\mathcal{L}^*_\sigma$ to the algebra generated by polynomials in the annihilation and creation operators.

Remark 3.4. Duality allows us to define by the relation

$$\mathrm{Tr}(\widehat{S}^*_k(\widehat{A})\ \rho\otimes\rho_k) := \mathrm{Tr}(\widehat{A}\ \widehat{S}_k(\rho\otimes\rho_k))\ , \tag{3.10}$$

the operator $\widehat{S}^*_k$ with domain consisting of operators $\widehat{A} = A\otimes a \in \mathfrak{A}(\mathcal{H}_\mathcal{C})\otimes\mathfrak{A}(\mathcal{H}_{\mathcal{A}_k})$. By definition (2.1) and by cyclicity of the trace, one obtains from (3.10) that $\widehat{S}^*_k = \widehat{S}^{-1}_k$, cf (2.7), (2.8). Then (3.3) and (3.10) yield for the adjoint of the operator defined in (3.3) the following expression

$$\widehat{L}^*_{\sigma,k}(\cdot) = \widehat{S}_k(L^*_{\sigma,k}(\widehat{S}^{-1}_k(\cdot)))\ . \tag{3.11}$$

Here $L^*_{\sigma,k}$ is defined in (1.43). It can be explicitly calculated on operators $A\otimes a$ with help of (1.38). Since (3.3) and (3.11) imply

$$\widehat{L}^*_{\sigma,k}(A\otimes a) = L^*_{\lambda,\sigma}(A)\otimes\eta_k a + L^*_{0,\sigma}(A)\otimes(I-\eta_k)a\ , \tag{3.12}$$

this equation gives an alternative way to establish (3.8) and (3.9). Similar to (3.4), the representation (3.12) indicates that for each $k \geq 1$ the mapping $\{e^{\tau \widehat{L}^*_{\sigma,k}}\}$ is reducible by two sub-algebras of $\mathfrak{A}(\mathcal{H}_\mathcal{C}) \otimes \mathfrak{A}(\mathcal{H}_{\mathcal{A}_k})$:

$$e^{\tau \widehat{L}^*_{\sigma,k}} : A \otimes \eta_k a \mapsto e^{\tau L^*_{\lambda,\sigma}}(A) \otimes \eta_k a \ , \tag{3.13}$$

$$e^{\tau \widehat{L}^*_{\sigma,k}} : A \otimes (I - \eta_k) a \mapsto e^{\tau L^*_{0,\sigma}}(A) \otimes (I - \eta_k) a \ . \tag{3.14}$$

Proof. (of Theorem 1.6) Using the Baker-Campbell-Hausdorff formula one can rewrite the Weyl operator in the form

$$W(\zeta) = e^{i(\overline{\zeta} b + \zeta b^*)/\sqrt{2}} = e^{i\zeta b^*/\sqrt{2}} e^{i\overline{\zeta} b/\sqrt{2}} e^{-|\zeta|^2/4} \ . \tag{3.15}$$

From the CCR we find that for any $\beta, \gamma \in \mathbb{C}$

$$e^{\beta b^*} b = (b - \beta) e^{\beta b^*} \ \text{and} \ \ e^{\gamma b} b^* = (b^* + \gamma) e^{\gamma b}. \tag{3.16}$$

By (3.15) and (3.16) the action of $L^*_{\lambda,\sigma}$ is

$$L^*_{\lambda,\sigma}(W(\zeta)) = -\frac{i}{\sqrt{2}} \mu \overline{\zeta}\, W(\zeta)\, b - \frac{i}{\sqrt{2}} \overline{\mu}\, \zeta\, b^* W(\zeta) - \frac{\sigma_+}{2} |\zeta|^2 W(\zeta) \tag{3.17}$$
$$+ \frac{i}{\sqrt{2}} \frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon} (\zeta + \overline{\zeta}) W(\zeta) \ .$$

Recall that $\mu = i\epsilon + (\sigma_- - \sigma_+)/2$. Therefore, the mapping generated by (3.8) (and correspondingly by (3.9)):

$$\gamma_{\lambda,\tau} := e^{\tau L^*_{\lambda,\sigma}} \tag{3.18}$$

is *quasi-free* [DVV1, DVV2]. In particular, there exist functions $\zeta(\tau)$ and $\Omega_{\lambda,\tau}(\zeta)$ such that

$$\gamma_{\lambda,\tau}(W(\zeta)) = e^{-\Omega_{\lambda,\tau}(\zeta)} W(\zeta(\tau)). \tag{3.19}$$

See Appendix A.4 for definitions and details.

To check this claim we calculate explicitly $\zeta(\tau)$ and $\Omega_{\lambda,\tau}(\zeta)$ using differential equation for (3.19):

$$\frac{d\gamma_{\lambda,\tau}(W(\zeta))}{d\tau} = L^*_{\lambda,\sigma}(\gamma_{\lambda,\tau}(W(\zeta))). \tag{3.20}$$

By (3.17) the right-hand side of (3.20) is given by

$$L^*_{\lambda,\sigma}(\gamma_{\lambda,\tau}(W(\zeta)))$$
$$= e^{-\Omega_{\lambda,\tau}(\zeta)} (-\frac{i}{\sqrt{2}} \mu \overline{\zeta(\tau)} W(\zeta(\tau)) b - \frac{i}{\sqrt{2}} \overline{\mu} \zeta(\tau) b^* W(\zeta(\tau))$$
$$- \frac{\sigma_+}{2} |\zeta(\tau)|^2 W(\zeta(\tau)) + \frac{i}{\sqrt{2}} \frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon} (\zeta(\tau) + \overline{\zeta(\tau)}) W(\zeta(\tau))). \tag{3.21}$$

The derivative of the τ-dependent Weyl operator $W(\zeta(\tau))$ is calculated using Baker-Campbell-Hausdorff formula (3.15)

$$\frac{dW(\zeta(\tau))}{d\tau} = \frac{i}{\sqrt{2}}\frac{d\zeta(\tau)}{d\tau}b^*W(\zeta(\tau)) + \frac{i}{\sqrt{2}}\frac{d\overline{\zeta(\tau)}}{d\tau}W(\zeta(\tau))b - \frac{1}{4}W(\zeta(\tau))\frac{d(\zeta(\tau)\overline{\zeta(\tau)})}{d\tau}.$$

Therefore, $\gamma_{\lambda,\tau}(W(\zeta))$ satisfies the following differential equation

$$\begin{aligned}\frac{d\gamma_{\lambda,\tau}(W(\zeta))}{d\tau} &= e^{-\Omega_{\lambda,\tau}(\zeta)}\left(\frac{i}{\sqrt{2}}\frac{d\zeta(\tau)}{d\tau}b^*W(\zeta(\tau)) + \frac{i}{\sqrt{2}}\frac{d\overline{\zeta(\tau)}}{d\tau}W(\zeta(\tau))b\right) \\ &+ e^{-\Omega_{\lambda,\tau}(\zeta)}\left(-\frac{d\Omega_{\lambda,\tau}(\zeta)}{d\tau}W(\zeta(\tau)) - \frac{1}{4}W(\zeta(\tau))\frac{d(\zeta(\tau)\overline{\zeta(\tau)})}{d\tau}\right).\end{aligned} \tag{3.22}$$

Due to (3.20), we can match the right-hand side of (3.22) with (3.21) and obtain the following system of differential equations for the functions $\zeta(\tau)$ and $\Omega_{\lambda,\tau}(\zeta)$:

$$\frac{d\zeta(\tau)}{d\tau} = -\overline{\mu}\zeta(\tau)$$

and

$$\frac{d\Omega_{\lambda,\tau}(\zeta)}{d\tau} = -\frac{1}{4}\frac{d(\zeta(\tau)\overline{\zeta(\tau)})}{d\tau} + \frac{\sigma_+}{2}|\zeta(\tau)|^2 - \frac{i}{\sqrt{2}}\frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon}(\zeta(\tau) + \overline{\zeta(\tau)}). \tag{3.23}$$

The solution to the first differential equation is

$$\zeta(\tau) = e^{-\overline{\mu}\tau}\zeta,$$

and using it in the second equation we find

$$\frac{d\Omega_{\lambda,\tau}(\zeta)}{d\tau} = \frac{\sigma_- + \sigma_+}{4}e^{-(\sigma_- - \sigma_+)\tau}|\zeta|^2 - \frac{i}{\sqrt{2}}\frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon}(e^{-\overline{\mu}\tau}\zeta + e^{-\mu\tau}\overline{\zeta}).$$

Therefore, the solution of (3.23) is

$$\begin{aligned}\Omega_{\lambda,\tau}(\zeta) &= \frac{|\zeta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}(1 - e^{-(\sigma_- - \sigma_+)\tau}) \\ &\quad - \frac{i}{\sqrt{2}}\frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon\mu}(1 - e^{-\mu\tau})\overline{\zeta} - \frac{i}{\sqrt{2}}\frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon\overline{\mu}}(1 - e^{-\overline{\mu}\tau})\zeta.\end{aligned}$$

Combining the solutions $\zeta(\tau)$ and $\Omega_{\lambda,\tau}(\zeta)$ with the expression for $\gamma_{\lambda,\tau}(W(\zeta))$ (3.19) we get

$$\begin{aligned}\gamma_{\lambda,\tau}(W(\zeta)) &= W(e^{-\overline{\mu}\tau}\zeta)\exp\left(-\frac{|\zeta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}(1 - e^{-(\sigma_- - \sigma_+)\tau})\right) \\ &\times \exp\left(\frac{i}{\sqrt{2}}\frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon\mu}(1 - e^{-\mu\tau})\overline{\zeta} + \frac{i}{\sqrt{2}}\frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon\overline{\mu}}(1 - e^{-\overline{\mu}\tau})\zeta\right).\end{aligned}$$

To calculate $\mathcal{L}^*_\sigma(W(\zeta))$ we use (3.6). Then for the first term one gets

$$\begin{aligned} & p\ S^{-1}(\gamma_{\lambda,\tau}(S(W(\zeta))) \\ &= p\ e^{-i\lambda(\zeta+\overline{\zeta})/\sqrt{2}\epsilon}\ e^{-\lambda(b^*-b)/\epsilon}\gamma_{\lambda,\tau}(W(\zeta))e^{\lambda(b^*-b)/\epsilon} \\ &= p\ \exp\left(-\frac{|\zeta|^2}{4}\frac{\sigma_-+\sigma_+}{\sigma_--\sigma_+}(1-e^{-(\sigma_--\sigma_+)\tau})\right) \\ &\times \exp\left(\frac{1}{\sqrt{2}}\left(\frac{\lambda}{\mu}(1-e^{-\mu\tau})\overline{\zeta}-\frac{\lambda}{\overline{\mu}}(1-e^{-\overline{\mu}\tau})\zeta\right)\right)W(e^{-\overline{\mu}\tau}\zeta)\ . \end{aligned}$$

If we put $\lambda = 0$ in this expression and substitute p for $1-p$, then we obtain the second term of (3.6). Together this yields the one-step evolution for the Weyl operator:

$$\begin{aligned} \mathcal{L}^*_\sigma(W(\zeta)) &= W(e^{-\overline{\mu}\tau}\zeta)\ \exp\left(-\frac{|\zeta|^2}{4}\frac{\sigma_-+\sigma_+}{\sigma_--\sigma_+}(1-e^{-(\sigma_--\sigma_+)\tau})\right) \\ &\times \left\{p\exp\left(\frac{1}{\sqrt{2}}\left(\frac{\lambda}{\mu}(1-e^{-\mu\tau})\overline{\zeta}-\frac{\lambda}{\overline{\mu}}(1-e^{-\overline{\mu}\tau})\zeta\right)\right)+1-p\right\}\ . \end{aligned} \tag{3.24}$$

Note that operator $\mathcal{L}^*_\sigma$ is a convex combination (3.6) of quasi-free (completely positive) maps and hence, in general, it is *not* quasi-free itself. Evolution of the Weyl operator for the time $t = n\tau$ (n-step evolution) follows from (3.24) by induction:

$$\begin{aligned} (\mathcal{L}^*_\sigma)^n(W(\zeta)) &= W(e^{-n\overline{\mu}\tau}\zeta)\exp\left(-\frac{|\zeta|^2}{4}\frac{\sigma_-+\sigma_+}{\sigma_--\sigma_+}(1-e^{-n(\sigma_--\sigma_+)\tau})\right) \\ &\times\prod_{k=0}^{n-1}\left\{p\exp\left(\frac{1}{\sqrt{2}}\left(\frac{\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\overline{\zeta}-\frac{\lambda}{\overline{\mu}}(1-e^{-\overline{\mu}\tau})e^{-k\overline{\mu}\tau}\zeta\right)\right)+1-p\right\}. \end{aligned} \tag{3.25}$$

If $\sigma_- > \sigma_+$, then $\mathrm{Re}(\mu) < 0$, which implies

$$w^* - \lim_{n\to\infty} W(e^{-n\overline{\mu}\tau}\zeta) = \mathbb{1}\ , \tag{3.26}$$

see Appendix A.3. To prove the convergence of the product (3.25) for the limit $n \to \infty$, we denote

$$h_k(\zeta) := p(\exp\frac{1}{\sqrt{2}}\left(\frac{\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\overline{\zeta}-\frac{\lambda}{\overline{\mu}}(1-e^{-\overline{\mu}\tau})e^{-k\overline{\mu}\tau}\zeta\right)-1)\ .$$

Since the product $\prod_{k=0}^{\infty}(1+h_k(\zeta))$ converges if and only if we establish convergence of the series $\sum_{k=0}^{\infty}|h_k(\zeta)|$, we have to estimate the terms $\{|h_k(\zeta)|\}_{k\geq 1}$. Note that

$$|h_k(\zeta)| \leq 2\sqrt{2}p\frac{\lambda|\zeta|}{|\mu|^2}(\frac{\sigma_--\sigma_+}{2}+\epsilon)(1+e^{-(\sigma_--\sigma_+)\tau/2})e^{-k(\sigma_--\sigma_+)\tau/2}\ . \tag{3.27}$$

Hence, (3.27) ensures the convergence of the series and the infinite product for $\sigma_- - \sigma_+ > 0$.

Summarizing, we obtain that for the open cavity with parameters $\sigma_- > \sigma_+ \geq 0$ the characteristic functional (see Appendix A.5) for the limiting state

$\omega_{C,\sigma}(\cdot)$ (1.50) exists and is given by

$$\omega_{C,\sigma}(W(\zeta)) = \exp\left(-\frac{|\zeta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\right) \tag{3.28}$$
$$\times \prod_{k=0}^{\infty}\left\{p\exp\left(\frac{1}{\sqrt{2}}\left(\frac{\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\overline{\zeta} - \frac{\lambda}{\overline{\mu}}(1-e^{-\overline{\mu}\tau})e^{-k\overline{\mu}\tau}\zeta\right)\right) + 1 - p\right\},$$

which is independent of the initial cavity state $\rho_{\mathcal{C}}$. □

Remark 3.5. Notice that the evolution of the Weyl operator for the time $t = n\tau$ can be written as a convex linear combination of quasi-free completely positive maps (see Appendix A.4):

$$(\mathcal{L}_\sigma^*)^n(W(\zeta)) = W(e^{-n\overline{\mu}\tau}\zeta)\exp\left(-\frac{|\zeta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}(1-e^{-n(\sigma_- - \sigma_+)\tau})\right)$$
$$\times \sum_{m=0}^{n-1} p^m(1-p)^{n-1-m} \sum_{1\le k_1<\ldots<k_m\le n-1}$$
$$\times \exp\left(\frac{1}{\sqrt{2}}\frac{\lambda}{|\mu|^2}\left(\overline{\mu}(1-e^{-\mu\tau})(e^{-k_1\mu\tau} + \ldots + e^{-k_m\mu\tau})\overline{\zeta} - \text{c.c.}\right)\right).$$

Here c.c. stands for the complex conjugate of the first term. Then (3.28) and the last formula suggest that the limiting state $\omega_{C,\sigma}(\cdot)$ is *not* quasi-free.

The rest of the chapter is devoted to the proof of Theorem 1.8, which is similar to the case $\sigma_\mp = 0$. The number of photons (1.55) for the time $t = n\tau$ can be calculated using the adjoint operator

$$N_\sigma(n\tau) = \mathrm{Tr}_{\mathcal{C}}(b^*b\ \mathcal{L}_\sigma^n(\rho_{\mathcal{C}})) = \mathrm{Tr}_{\mathcal{C}}((\mathcal{L}_\sigma^*)^n(b^*b)\ \rho_{\mathcal{C}})\ , \tag{3.29}$$

where $\rho_{\mathcal{C}}$ is the initial gauge-invariant state of the cavity. Note that by (3.6) and (3.7) we obtain for $A = b^*b$

$$\mathcal{L}_\sigma^*(b^*b) = p\ S^{-1}(e^{\tau L_{\lambda,\sigma}^*}S(b^*b)) + (1-p)e^{\tau L_{0,\sigma}^*}(b^*b)\ . \tag{3.30}$$

Lemma 3.6. *The action of the adjoint operator $\mathcal{L}_\sigma^*$ on the photon number operator is explicitly given by*

$$\mathcal{L}_\sigma^*(b^*b) \tag{3.31}$$
$$= e^{-(\sigma_- - \sigma_+)\tau}b^*b + p\frac{i\lambda}{\mu}e^{-(\sigma_- - \sigma_+)\tau}(1-e^{\mu\tau})b^* - p\frac{i\lambda}{\overline{\mu}}e^{-(\sigma_- - \sigma_+)\tau}(1-e^{\overline{\mu}\tau})b$$
$$+ p\frac{\lambda^2}{|\mu|^2}e^{-(\sigma_- - \sigma_+)\tau}(1-e^{\mu\tau})(1-e^{\overline{\mu}\tau}) + \frac{\sigma_+}{\sigma_- - \sigma_+}(1-e^{-(\sigma_- - \sigma_+)\tau}),$$

where $\mu = i\epsilon + (\sigma_- - \sigma_+)/2$.

Proof. We start with the first term in the right-hand side of (3.30). Since $\gamma_{\lambda,\tau}(A) = e^{\tau L_{\lambda,\sigma}^*}(A)$ (3.18), one can calculate $\gamma_{\lambda,\tau}(S(b^*b))$ by taking into account (3.8). Then

$$L_{\lambda,\sigma}^*((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)) = i\lambda b - i\lambda b^* - (\sigma_- - \sigma_+)(b^* - \lambda/\epsilon)(b - \lambda/\epsilon) + \sigma_+$$

and

$$L^*_{\lambda,\sigma}(b^*) = -\overline{\mu} b^* + \frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon} \ , \ L^*_{\lambda,\sigma}(b) = -\mu b + \frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon} \ .$$

Therefore, we obtain for mapping $\gamma_{\lambda,\tau}(\cdot)$ the following system of differential equations:

$$\begin{aligned}
&\frac{d\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)}{d\tau} \\
&= -(\sigma_- - \sigma_+)\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)) + i\lambda\gamma_{\lambda,\tau}(b) - i\lambda\gamma_{\lambda,\tau}(b^*) + \sigma_+ \\
&\frac{d\gamma_{\lambda,\tau}(b)}{d\tau} = -\mu\gamma_{\lambda,\tau}(b) + \frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon} \\
&\frac{d\gamma_{\lambda,\tau}(b^*)}{d\tau} = -\overline{\mu}\gamma_{\lambda,\tau}(b^*) + \frac{\lambda(\sigma_- - \sigma_+)}{2\epsilon}.
\end{aligned}$$

The solution of this system is

$$\gamma_{\lambda,\tau}(b^*) = e^{-\overline{\mu}\tau} b^* + \frac{\lambda}{\epsilon}(1 - e^{-\overline{\mu}\tau}) + i \ \frac{\lambda}{\overline{\mu}}(1 - e^{-\overline{\mu}\tau}) \ , \tag{3.32}$$

$$\gamma_{\lambda,\tau}(b) = e^{-\mu\tau} b + \frac{\lambda}{\epsilon}(1 - e^{-\mu\tau}) - i \ \frac{\lambda}{\mu}(1 - e^{-\mu\tau}) \ , \tag{3.33}$$

$$\begin{aligned}
&\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)) = e^{-(\sigma_- - \sigma_+)\tau} b^* b + (\frac{i\lambda}{\mu}(1 - e^{\mu\tau}) - \frac{\lambda}{\epsilon})e^{-(\sigma_- - \sigma_+)\tau} b^* \\
&+ (-\frac{i\lambda}{\overline{\mu}}(1 - e^{\overline{\mu}\tau}) - \frac{\lambda}{\epsilon})e^{-(\sigma_- - \sigma_+)\tau} b + \frac{\lambda^2}{|\mu|^2}(1 - e^{-(\sigma_- - \sigma_+)\tau}) \\
&+ \frac{\sigma_+}{\sigma_- - \sigma_+}(1 - e^{-(\sigma_- - \sigma_+)\tau}) + \frac{\lambda^2}{\epsilon^2} e^{-(\sigma_- - \sigma_+)\tau} \\
&- \frac{\lambda^2(\sigma_- - \sigma_+)\sin\epsilon\tau}{\epsilon|\mu|^2} e^{-(\sigma_- - \sigma_+)\tau/2} \ .
\end{aligned} \tag{3.34}$$

Making the shift transformation (2.1) of $\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon))$ and calculating the second term in (3.30) by setting $\lambda = 0$, we obtain (3.31). □

If one plugs in (3.6) and in (3.8),(3.9) operators $A = b^*$ or $= b$, then (3.32), (3.33) yield for $k \geq 1$

$$(\mathcal{L}^*_\sigma)^k(b^*) = e^{-k\bar{\mu}\tau} b^* + p\frac{i\lambda}{\bar{\mu}}(1 - e^{-k\bar{\mu}\tau}) \ , \tag{3.35}$$

$$(\mathcal{L}^*_\sigma)^k(b) = e^{-k\mu\tau} b - p\frac{i\lambda}{\mu}(1 - e^{-k\mu\tau}) \ . \tag{3.36}$$

Since $\mathcal{L}^*_\sigma(\cdot) \mid_{\sigma_- = \sigma_+ = 0} = \mathcal{L}^*(\cdot)$, formulae (3.35), (3.36) coincide for $\sigma_- = \sigma_+ = 0$ with (2.19), (2.20) for the ideal cavity.

Notice also that (3.6),(3.7) and (3.8),(3.9) imply

$$(\mathcal{L}^*_\sigma)(\mathbb{1}) = 0 \ . \tag{3.37}$$

Proof. (of Theorem 1.8) To this end, we construct first a 4×4 matrix $\widehat{\mathcal{L}}^*_\sigma$ acting on the complex linear space spanned by the operator-valued vectors $(b^*b, 0, 0, 0)$, $(0, b^*, 0, 0)$, $(0, 0, b, 0)$, $(0, 0, 0, \mathbb{1})$), according to the formulae (3.31), (3.35), (3.36) and (3.37), for $k = 1$. Then diagonalisation of $\widehat{\mathcal{L}}^*_\sigma$ allows to calculate powers $(\widehat{\mathcal{L}}^*_\sigma)^n$ and to find explicit expressions for the n-step mapping

$$\begin{aligned}
(\mathcal{L}^*_\sigma)^n(b^*b) &= e^{-n(\sigma_- - \sigma_+)\tau} b^*b + p\frac{i\lambda}{\mu}(e^{-n(\sigma_- - \sigma_+)\tau} - e^{-n\bar{\mu}\tau})b^* \qquad (3.38)\\
&- p\frac{i\lambda}{\bar{\mu}}(e^{-n(\sigma_- - \sigma_+)\tau} - e^{-n\mu\tau})b \\
&+ p\frac{\lambda^2}{|\mu|^2} e^{-(\sigma_- - \sigma_+)\tau}(1 - e^{\mu\tau})(1 - e^{\bar{\mu}\tau})\frac{1 - e^{-n(\sigma_- - \sigma_+)\tau}}{1 - e^{-(\sigma_- - \sigma_+)\tau}} \\
&- p^2\frac{2\lambda^2}{|\mu|^2}\frac{1 - e^{-n(\sigma_- - \sigma_+)\tau}}{1 - e^{-(\sigma_- - \sigma_+)\tau}}(1 - e^{-\frac{\sigma_- - \sigma_+}{2}\tau}\cos\epsilon\tau) \\
&+ p^2\frac{2\lambda^2}{|\mu|^2}(1 - e^{-n(\sigma_- - \sigma_+)\tau/2}\cos n\epsilon\tau) + \frac{\sigma_+}{\sigma_- - \sigma_+}(1 - e^{-n(\sigma_- - \sigma_+)\tau}).
\end{aligned}$$

Note that (3.38) reduces to (3.31) for $n = 1$. If $\tau = 0$, then (3.38) yields b^*b for any n. By (3.38) we obtain for a *gauge-invariant* initial state the mean-value of the photon number (1.55) in the open cavity at $t = n\tau$:

$$\begin{aligned}
N_\sigma(n\tau) &:= \mathrm{Tr}_\mathcal{C}(\rho_C(\mathcal{L}^*_\sigma)^n(b^*b)) = e^{-n(\sigma_- - \sigma_+)\tau} N_\sigma(0) \qquad (3.39)\\
&+ p(1-p)\frac{2\lambda^2}{|\mu|^2}(1 - e^{-(\sigma_- - \sigma_+)\tau/2}\cos\epsilon\tau)\frac{1 - e^{-n(\sigma_- - \sigma_+)\tau}}{1 - e^{-(\sigma_- - \sigma_+)\tau}} \\
&+ p^2\frac{2\lambda^2}{|\mu|^2}(1 - e^{-n(\sigma_- - \sigma_+)\tau/2}\cos n\epsilon\tau) + \frac{\sigma_+}{\sigma_- - \sigma_+}(1 - e^{-n(\sigma_- - \sigma_+)\tau})\ .
\end{aligned}$$

Note that (3.39) coincides with (1.57) for $t = n\tau$. Finally, by virtue of (3.38) and (3.39) one gets for the w^*-limit (see Appendix A.3)

$$\begin{aligned}
&w^* - \lim_{n\to\infty}(\mathcal{L}^*_\sigma)^n(b^*b) \qquad (3.40)\\
&= p(1-p)\frac{2\lambda^2}{|\mu|^2}\frac{1 - e^{-(\sigma_- - \sigma_+)\tau/2}\cos\epsilon\tau}{1 - e^{-(\sigma_- - \sigma_+)\tau}} + p(2p-1)\frac{\lambda^2}{|\mu|^2} + \frac{\sigma_+}{\sigma_- - \sigma_+}\ .
\end{aligned}$$

Therefore, $\lim_{n\to\infty}\omega^{n\tau}_{\mathcal{C},\sigma}(b^*b) = \lim_{n\to\infty}\mathrm{Tr}_\mathcal{C}(\rho_C(\mathcal{L}^*_\sigma)^n(b^*b))$ and (3.40) yield the proof of (1.58). $\square$

Notice that in the limit of the ideal cavity: $\sigma_- \to +0$, $\sigma_- > \sigma_+ \geq 0$, (3.39) gives $\lim_{\sigma_-\to+0} N_\sigma(t) = N(t)$, which is (1.28). For other limiting cases see Remark 1.9.

4. Energy and Entropy Variations

4.1. Energy variation in the ideal cavity

Since the time-dependent interaction in (1.3) is piecewise constant, the system is autonomous on each interval $[(n-1)\tau, n\tau)$. Therefore, there is no variation of energy on this interval, but it may *jump*, when a new atom enters into the cavity. Note that although the *total* energy corresponding to the infinite system (1.5) is undefined its variation is well-defined [BJM1]-[BJM3].

The times when the n-th atom is actually traveling in the cavity are of the form $t = n(t)\tau + \nu(t)$, with $n(t) = n-1$ and $\nu(t) \in [0, \tau)$, see (1.15). To calculate the energy variation we compare two total energy expectations: for the moment $t_n = (n-1)\tau + \nu(t_n)$, when the n-th atom is present in the cavity, and for the moment $t_{n-1} = (n-2)\tau + \nu(t_{n-1})$, when the $(n-1)$-th atom was in the cavity. Then by (1.5), (1.6) and by (1.16) we obtain for the total energy variation in the system (1.5) between two moments t_{n-1} and t_n the following expression:

$$\begin{aligned}
\Delta\mathcal{E}(t_n, t_{n-1}) &:= \omega_{\mathcal{S}}^{t_n}(H(t_n)) - \omega_{\mathcal{S}}^{t_{n-1}}(H(t_{n-1})) \\
&= \mathrm{Tr}(e^{-i\nu(t_n)H_n}\rho_{\mathcal{S}}((n-1)\tau)e^{-i\nu(t_n)H_n}H_n) \\
&- \mathrm{Tr}(e^{-i\nu(t_{n-1})H_{n-1}}\rho_{\mathcal{S}}((n-2)\tau)e^{i\nu(t_{n-1})H_{n-1}}H_{n-1}) \\
&= \mathrm{Tr}(\rho_{\mathcal{S}}((n-1)\tau)H_n) \\
&- \mathrm{Tr}(e^{-i\tau H_{n-1}}\rho_{\mathcal{S}}((n-2)\tau)e^{i\tau H_{n-1}}H_{n-1}) \\
&= \mathrm{Tr}(T_{(n-1)\tau,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})[H_n - H_{n-1}])\,.
\end{aligned} \tag{4.1}$$

Here we used that the system (1.5), (1.6) is piecewise autonomous with $H(t+0) = H(t)$, and the state $\rho_{\mathcal{S}}(t)$ is w^*-time-continuous (Appendix A.2).

Recall that by duality (1.34) we have

$$\begin{aligned}
&\mathrm{Tr}\,(T_{(n-1)\tau,0}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})[H_n - H_{n-1}]) \\
&= \mathrm{Tr}\,(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}\; T^*_{(n-1)\tau,0}(H_n - H_{n-1}))\ .
\end{aligned} \tag{4.2}$$

Since (1.6) implies

$$H_n - H_{n-1} = \lambda\,(b^* + b) \otimes (\eta_n - \eta_{n-1})\ , \tag{4.3}$$

by (4.1), (4.2) and by $[H_{k'}, \eta_k] = 0$ we obtain

$$\begin{aligned}
&\Delta\mathcal{E}(t_n, t_{n-1}) \\
&= \mathrm{Tr}\{\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}\; T^*_{(n-1)\tau,0}(\lambda\,(b^* + b) \otimes \mathbb{1})[\mathbb{1} \otimes (\eta_n - \eta_{n-1})]\}
\end{aligned} \tag{4.4}$$

Lemma 4.1. *For any $n \geq 1$ one gets:*

$$T^*_{n\tau,0}(b^* \otimes \mathbb{1}) = e^{ni\tau\epsilon}b^* \otimes \mathbb{1} - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon})\sum_{k=1}^{n} e^{(n-k)i\tau\epsilon}\mathbb{1} \otimes \eta_k\ , \tag{4.5}$$

$$T^*_{n\tau,0}(b \otimes \mathbb{1}) = e^{-ni\tau\epsilon}b \otimes \mathbb{1} - \frac{\lambda}{\epsilon}(1 - e^{-i\tau\epsilon})\sum_{k=1}^{n} e^{-(n-k)i\tau\epsilon}\mathbb{1} \otimes \eta_k\ . \tag{4.6}$$

Proof. Let us define

$$B_k^*(\tau) := T_k^*(b^* \otimes \mathbb{1}) = e^{i\tau H_k}(b^* \otimes \mathbb{1})e^{-i\tau H_k} \ , \ k \geq 1 \ . \tag{4.7}$$

Then by (1.6) and (1.32), (1.33) the operator (4.7) is solution of equation

$$\partial_s B_k^*(s) = i[H_k, B_k^*(s)] = i\epsilon B_k^*(s) + \lambda \mathbb{1} \otimes \eta_k \ , \ B_k^*(0) = b^* \otimes \mathbb{1} \ ,$$

which has the following explicit form:

$$B_k^*(\tau) = e^{i\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon})\mathbb{1} \otimes \eta_k \ . \tag{4.8}$$

Similarly one obtains

$$B_k(\tau) = e^{-i\tau\epsilon}(b \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{-i\tau\epsilon})\mathbb{1} \otimes \eta_k \ . \tag{4.9}$$

Note that by iteration of (4.8) for $k = 1, 2$ one gets

$$\begin{aligned} T_{2\tau,0}^*(b^* \otimes \mathbb{1}) = e^{2i\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon})e^{i\tau\epsilon}\mathbb{1} \otimes \eta_1 \\ - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon})\mathbb{1} \otimes \eta_2 \ . \end{aligned} \tag{4.10}$$

Proceeding with iteration of (4.10) we obtain (4.5). Using (4.9) one proves in a similar way (4.6). □

Recall that in the present paper we suppose that atomic beam is homogeneous (1.2), i.e. $p = \mathrm{Tr}\{\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}(\mathbb{1} \otimes \eta_n)\}$ is independent of n probability that the n-th atom is in the excited state, and that the atomic beam is Bernoulli (Remark 1.1):

$$\mathrm{Tr}_{\mathcal{A}}\{\rho_{\mathcal{A}}(\eta_{n_1}\eta_{n_2})\} = \delta_{n_1,n_2}\, p + (1 - \delta_{n_1,n_2})\, p^2 \ . \tag{4.11}$$

Let the initial cavity state $\rho_{\mathcal{C}}$ be gauge-invariant. Then by Lemma 4.1 and (4.11) one obtains:

$$\begin{aligned} &\mathrm{Tr}\{\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}\ T_{(n-1)\tau,0}^*(\lambda\,(b^* + b) \otimes \mathbb{1})[\mathbb{1} \otimes (\eta_n - \eta_{n-1})]\} \\ &= -\frac{\lambda^2}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^{n-1} e^{(n-k-1)i\tau\epsilon} \mathrm{Tr}_{\mathcal{A}}\rho_{\mathcal{A}}\eta_k(\eta_n - \eta_{n-1}) \\ &- \frac{\lambda^2}{\epsilon}(1 - e^{-i\tau\epsilon}) \sum_{k=1}^{n-1} e^{-(n-k-1)i\tau\epsilon} \mathrm{Tr}_{\mathcal{A}}\rho_{\mathcal{A}}\eta_k(\eta_n - \eta_{n-1}) \\ &= p(1-p)\ \frac{2\lambda^2}{\epsilon}\ (1 - \cos\tau\epsilon) \ . \end{aligned} \tag{4.12}$$

Hence, formulae (4.4), (4.12) prove for the energy variation on the interval (t_{n-1}, t_n) the following statement.

Theorem 4.2. *For the case of the ideal cavity the total energy variation* (4.1) *between two moments t_{n-1} and t_n, where $n \geq 1$ is*

$$\Delta\mathcal{E}(t_n, t_{n-1}) = p(1-p)\ \frac{2\lambda^2}{\epsilon}(1 - \cos\tau\epsilon) \ , \tag{4.13}$$

i.e. for the variation between $t_0 = \nu(t_0)$ and $t_n \geq t_0$ we obtain:

$$\Delta\mathcal{E}(t_n, t_0) = \sum_{k=1}^{n} \Delta\mathcal{E}(t_k, t_{k-1}) = (n-1)\ p(1-p)\ \frac{2\lambda^2}{\epsilon}(1 - \cos\tau\epsilon)\ . \quad (4.14)$$

Remark 4.3. The total energy variation (4.1), when the n-th atom is traveling through the cavity between the moments $t' = (n-1)\tau$ and $t'' = n\tau - 0$, can be written as

$$\Delta\mathcal{E}(t'', t') = \omega_{\mathcal{S}}^{n\tau}(H_n) - \omega_{\mathcal{S}}^{(n-1)\tau}(H_n)\ . \quad (4.15)$$

Here again we used that for $t \in [(n-1)\tau, n\tau)$ the Hamiltonian (1.5) is piecewise *constant* and that it has the form (1.6), as well as that the state $\omega_{\mathcal{S}}^{t}(\cdot)$ is w^*-continuous in time (Appendix A.2). Since the system (1.5) is autonomous on the interval $[(n-1)\tau, n\tau)$, one obtains $\Delta\mathcal{E}(n\tau - 0, (n-1)\tau) = 0$. Then (4.15) implies that on this interval (in contrast to (4.4)) the variation of the interaction-energy *completely* compensates the energy variation due to the photon number pumping:

$$\begin{aligned} &\omega_{\mathcal{S}}^{n\tau}(\lambda\,(b^* + b) \otimes \eta_n) - \omega_{\mathcal{S}}^{(n-1)\tau}(\lambda\,(b^* + b) \otimes \eta_{n-1}) \\ &= -[\omega_{\mathcal{S}}^{n\tau}(\epsilon b^* b \otimes \mathbb{1}) - \omega_{\mathcal{S}}^{(n-1)\tau}(\epsilon b^* b \otimes \mathbb{1})]\ . \end{aligned} \quad (4.16)$$

Note that similar to (4.12) one can check this identity explicitly using Lemma 4.1 and (4.11) applied to the left-hand side of (4.16).

4.2. Energy variation in the open cavity

Although for the open cavity the time-dependent generator (1.37) is still piecewise *constant* (1.38), the cavity energy is continuously varying between the moments $\{t = k\tau\}_{k\geq 0}$ (when the interaction may to jump (1.5)) because of the leaking/injection of photons.

Therefore, as above we first concentrate on the elementary variation of the total energy, when the n-th atom is traveling through the cavity between the moments $t' = (n-1)\tau$ and $t'' = n\tau - 0$:

$$\Delta\mathcal{E}_\sigma(t'', t') := \omega_{\mathcal{S},\sigma}^{n\tau}(H_n) - \omega_{\mathcal{S},\sigma}^{(n-1)\tau}(H_n)\ . \quad (4.17)$$

Here again we used two facts: (1) for $t \in [(n-1)\tau, n\tau)$ the Hamiltonian (1.5) of the form (1.6) is piecewise constant; (2) the state $\omega_{\mathcal{S},\sigma}^{t}(\cdot)$ (1.42) is time-continuous (Appendix A.2).

By virtue of (1.42) and of (1.43) for operator $A = H_n$ (1.6), we see that the problem (4.17) reduces to calculation of the following expectations:

$$\epsilon\ \omega_{\mathcal{S},\sigma}^{k\tau}(b^* b \otimes \mathbb{1}) \ \text{ and }\ \lambda\ \omega_{\mathcal{S},\sigma}^{s\tau}((b^* + b) \otimes \eta_k)\ ,\ k, s \geq 1\ . \quad (4.18)$$

The first expectation in (4.18) is known due to (1.46) and Theorem 1.8, see (3.39):

$$\epsilon\ \omega_{\mathcal{S},\sigma}^{k\tau}(b^* b \otimes \mathbb{1}) = \epsilon\, N_\sigma(k\tau)\ . \quad (4.19)$$

To calculate the second expectation in (4.18) we use (1.42) for the operator $A = ((b^* + b) \otimes \mathbb{1})(\mathbb{1} \otimes \eta_k)$ and the representation (1.43) for initial gauge-invariant state $\rho_{\mathcal{C}}$ and for homogeneous atoms state $\rho_{\mathcal{A}}$.

Lemma 4.4. *Let $\sigma_- > \sigma_+ \geq 0$. Then for the mappings $\{(T^{\sigma}_{n\tau,0})^*\}_{n\geq 0}$, see* (1.43)*, one obtains:*

$$(T^{\sigma}_{n\tau,0})^*(b\otimes \mathbb{1}) = e^{-n\mu\tau} b\otimes \mathbb{1} - \frac{\lambda i}{\mu}(1-e^{-\mu\tau})\sum_{k=1}^{n} e^{-(n-k)\mu\tau}\mathbb{1}\otimes\eta_k\ , \quad (4.20)$$

and $(T^{\sigma}_{n\tau,0})^*(b^*\otimes \mathbb{1}) = ((T^{\sigma}_{n\tau,0})^*(b\otimes \mathbb{1}))^*$.

Proof. We prove this lemma by induction. Suppose that formula (4.20) is true for $(T^{\sigma}_{n\tau,0})^*$. Then we show that it is also valid for $(T^{\sigma}_{(n+1)\tau,0})^*$. By virtue of (1.43) and by (3.4),(3.11) for the action of operator $e^{\tau L^*_{\sigma,n}}$, one gets

$$\begin{aligned}
&(T^{\sigma}_{n+1})^*(b\otimes \mathbb{1}) = (T^{\sigma}_{n})^*(e^{\tau L^*_{\sigma,n+1}}(b\otimes \mathbb{1}))\\
&= (T^{\sigma}_{n})^*(\widehat{S}^{-1}_{n+1}(e^{\tau \widehat{L}^*_{\sigma,n+1}}(\widehat{S}_{n+1}(b\otimes \mathbb{1}))))\\
&= (T^{\sigma}_{n})^*(\widehat{S}^{-1}_{n+1}(e^{\tau \widehat{L}^*_{\sigma,n+1}}\Big((b-\frac{\lambda}{\epsilon})\otimes\eta_{n+1} + b\otimes(I-\eta_{n+1})\Big)))\ ,
\end{aligned}$$

where we used (2.2) in the last line. By (3.13),(3.14) and (3.18),(3.33) combined with the shift $\widehat{S}^{-1}_{n+1}$ (2.8), we obtain

$$\begin{aligned}
&\widehat{S}^{-1}_{n+1}(e^{\tau \widehat{L}^*_{\sigma,n+1}}((b-\frac{\lambda}{\epsilon})\otimes\eta_{n+1})) = \widehat{S}^{-1}_{n+1}((\gamma_{\lambda,\tau}(b)-\frac{\lambda}{\epsilon})\otimes\eta_{n+1})\\
&= (e^{-\mu\tau}b - \frac{\lambda i}{\mu}(1-e^{-\mu\tau}))\otimes\eta_{n+1}\ ,\\
&\widehat{S}^{-1}_{n+1}(e^{\tau \widehat{L}^*_{\sigma,n+1}}(b\otimes(I-\eta_{n+1}))) = \widehat{S}^{-1}_{n+1}(\gamma_{0,\tau}(b)\otimes(I-\eta_{n+1}))\\
&= e^{-\mu\tau}b\otimes(I-\eta_{n+1})\ .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(T^{\sigma}_{n+1})^*(b\otimes \mathbb{1}))\\
&= (T^{\sigma}_{n})^*((e^{-\mu\tau}b - \frac{\lambda i}{\mu}(1-e^{-\mu\tau}))\otimes\eta_{n+1} + e^{-\mu\tau}b\otimes(\mathbb{1}-\eta_{n+1}))\\
&= (T^{\sigma}_{n})^*(e^{-\mu\tau}b\otimes\mathbb{1} - \frac{\lambda i}{\mu}(1-e^{-\mu\tau})\mathbb{1}\otimes\eta_{n+1})\\
&= e^{-(n+1)\mu\tau}b\otimes\mathbb{1} - \frac{\lambda i}{\mu}(1-e^{-\mu\tau})\sum_{k=1}^{n+1}e^{-(n-k+1)\mu\tau}\mathbb{1}\otimes\eta_k,
\end{aligned}$$

which proves the lemma. □

Recall that $\mu = i\epsilon + (\sigma_- - \sigma_+)/2$. Hence, in the limit $\sigma_- \to +0$ one recovers from this Lemma formulae (4.5) and (4.6) for the ideal cavity.

Corollary 4.5. Let initial cavity state $\rho_{\mathcal{C}}$ be gauge-invariant state for homogeneous state $\rho_{\mathcal{A}}$ of the atomic beam. Then with help of (4.11) and (4.20) one obtains the interaction energy expectations (4.18) corresponding to the difference (4.17):

$$\lambda\, \omega_{\mathcal{S},\sigma}^{n\tau}((b^* + b)\otimes \eta_n) = \tag{4.21}$$
$$- \frac{2\lambda^2\epsilon}{|\mu|^2}\left[p(1-p)(1-e^{-(\sigma_- -\sigma_+)\tau/2}\cos\epsilon\tau) + p^2(1-e^{-n(\sigma_- -\sigma_+)\tau/2}\cos n\epsilon\tau)\right]$$
$$+ \frac{\lambda^2(\sigma_- - \sigma_+)}{|\mu|^2}\left[p(1-p)e^{-(\sigma_- -\sigma_+)\tau/2}\sin\epsilon\tau + p^2 e^{-n(\sigma_- -\sigma_+)\tau/2}\sin n\epsilon\tau\right]\ ,$$

$$\lambda\, \omega_{\mathcal{S},\sigma}^{(n-1)\tau}((b^* + b)\otimes \eta_n) = -\frac{2\lambda^2\epsilon}{|\mu|^2}p^2(1-e^{-(n-1)(\sigma_- -\sigma_+)\tau/2}\cos(n-1)\epsilon\tau)$$
$$+ \frac{\lambda^2(\sigma_- - \sigma_+)}{|\mu|^2}p^2 e^{-(n-1)(\sigma_- -\sigma_+)\tau/2}\sin(n-1)\epsilon\tau\ . \tag{4.22}$$

Corollary 4.6. Taking into account Theorem 1.8 and (4.21), (4.22) we get for the elementary variation of the total energy (4.17)

$$\Delta\mathcal{E}_\sigma(n\tau - 0, (n-1)\tau) = \epsilon\,(N_\sigma(n\tau) - N_\sigma((n-1)\tau)) \tag{4.23}$$
$$+ \lambda\ (\omega_{\mathcal{S},\sigma}^{n\tau}((b^* + b)\otimes\eta_n) - \omega_{\mathcal{S},\sigma}^{(n-1)\tau}((b^* + b)\otimes\eta_n))$$
$$= \epsilon\,(\frac{\sigma_+}{\sigma_- - \sigma_+} - N_\sigma(0))(1-e^{-(\sigma_- -\sigma_+)\tau})e^{-(n-1)(\sigma_- -\sigma_+)\tau}$$
$$- p(1-p)\frac{2\lambda^2\epsilon}{|\mu|^2}(1-e^{-(\sigma_- -\sigma_+)\tau/2}\cos\epsilon\tau)(1-e^{-(n-1)(\sigma_- -\sigma_+)\tau})$$
$$+ p(1-p)\frac{\lambda^2(\sigma_- - \sigma_+)}{|\mu|^2}e^{-(\sigma_- -\sigma_+)\tau/2}\sin\epsilon\tau$$
$$+ p^2\frac{\lambda^2(\sigma_- - \sigma_+)}{|\mu|^2}\left[e^{-n(\sigma_- -\sigma_+)\tau/2}\sin n\epsilon\tau - e^{-(n-1)(\sigma_- -\sigma_+)\tau/2}\sin(n-1)\epsilon\tau\right].$$

Note that in the limit of the ideal cavity: $\sigma_+ \to 0$ and $\sigma_- \to 0$, one finds for total energy variation (4.23): $\Delta\mathcal{E}_\sigma(n\tau - 0, (n-1)\tau) = 0$, which corresponds to the autonomous case, see Remark 4.3. Whereas for $\sigma_- > \sigma_+ \geq 0$ the external pumping due to $\sigma_+/(\sigma_- - \sigma_+) \geq 0$ is in competition with the energy leaking, see the second term in the right-hand side of (4.23). The limit of the energy increment when $n \to \infty$ is

$$\lim_{n\to\infty}\Delta\mathcal{E}_\sigma(n\tau - 0, (n-1)\tau) = \tag{4.24}$$
$$p(1-p)\frac{\lambda^2}{|\mu|^2}\left[-2\epsilon(1-e^{-(\sigma_- -\sigma_+)\tau/2}\cos\epsilon\tau) + (\sigma_- - \sigma_+)e^{-(\sigma_- -\sigma_+)\tau/2}\sin\epsilon\tau\right].$$

Remark 4.7. To consider the impact when the n-th atom enters the cavity we study the total energy variation on the extended interval $((n-1)\tau - 0, n\tau - 0)$. Then

$$\Delta\mathcal{E}_\sigma(n\tau - 0, (n-1)\tau - 0) = (\omega_{\mathcal{S},\sigma}^{n\tau-0}(H_n) - \omega_{\mathcal{S},\sigma}^{(n-1)\tau}(H_n)) \tag{4.25}$$
$$+ (\omega_{\mathcal{S},\sigma}^{(n-1)\tau}(H_n) - \omega_{\mathcal{S},\sigma}^{(n-1)\tau-0}(H_{n-1}))\ ,$$

where the second difference $\Delta\mathcal{E}_\sigma((n-1)\tau,(n-1)\tau-0) := \omega_{\mathcal{S},\sigma}^{(n-1)\tau}(H_n) - \omega_{\mathcal{S},\sigma}^{(n-1)\tau-0}(H_{n-1})$ corresponds to the energy variation (*jump*), when the n-th atom enters the cavity and the $(n-1)$-th atom leaves it.

To calculate $\Delta\mathcal{E}_\sigma((n-1)\tau,(n-1)\tau-0)$ note that by the time continuity of the state

$$\begin{aligned}
&\Delta\mathcal{E}_\sigma((n-1)\tau,(n-1)\tau-0) = \mathrm{Tr}(\rho_{\mathcal{S}}((n-1)\tau)H_n) - \mathrm{Tr}(\rho_{\mathcal{S}}((n-1)\tau)H_{n-1})\\
&= \mathrm{Tr}\left(e^{\tau L_{\sigma,n-1}}\ ...\ e^{\tau L_{\sigma,1}}(\rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}}))(H_n-H_{n-1})\right)\\
&= \mathrm{Tr}\left(T^\sigma_{(n-1)\tau,0}(\rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}})(H_n-H_{n-1})\right)\\
&= \mathrm{Tr}\left(\rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}}(T^\sigma_{(n-1)\tau,0})^*(\lambda(b^*+b)\otimes(\eta_n-\eta_{n-1})\right)\\
&= \mathrm{Tr}\left(\rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}}(T^\sigma_{(n-1)\tau,0})^*(\lambda(b^*+b)\otimes(\eta_n-\eta_{n-1})\right)\\
&= \mathrm{Tr}\{\rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}}(T^\sigma_{(n-1)\tau,0})^*(\lambda(b^*+b)\otimes\mathbb{1})[\mathbb{1}\otimes(\eta_n-\eta_{n-1})]\},
\end{aligned}$$

where $T^\sigma_{t=n\tau,0} = e^{\tau L_{\sigma,n}}\ ...\ e^{\tau L_{\sigma,1}}$ is defined by (1.40).

If the initial cavity state is gauge-invariant, then (4.20) yields for the energy jump at the moment $t=(n-1)\tau$:

$$\begin{aligned}
&\Delta\mathcal{E}_\sigma((n-1)\tau,(n-1)\tau-0) = \\
&\mathrm{Tr}\{\rho_{\mathcal{C}}\otimes\rho_{\mathcal{A}}(T^\sigma_{(n-1)\tau,0})^*(\lambda(b^*+b)\otimes\mathbb{1})[\mathbb{1}\otimes(\eta_n-\eta_{n-1})]\}\\
&= \frac{\lambda^2 i}{\bar\mu}(1-e^{-\bar\mu\tau})\sum_{k=1}^{n-1}e^{-(n-k-1)\bar\mu\tau}\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}}\eta_k(\eta_n-\eta_{n-1}))\\
&- \frac{\lambda^2 i}{\mu}(1-e^{-\mu\tau})\sum_{k=1}^{n-1}e^{-(n-k-1)\mu\tau}\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}}\eta_k(\eta_n-\eta_{n-1})).
\end{aligned} \tag{4.26}$$

Taking into account the Bernoulli property (4.11) we obtain from (4.26)

$$\begin{aligned}
&\Delta\mathcal{E}_\sigma((n-1)\tau,(n-1)\tau-0) = \frac{\lambda^2 i}{\mu}(1-e^{-\mu\tau})p(1-p) - \frac{\lambda^2 i}{\bar\mu}(1-e^{-\bar\mu\tau})p(1-p)\\
&= p(1-p)\frac{2\lambda^2\epsilon}{|\mu|^2}(1-e^{-(\sigma_--\sigma_+)\tau/2}\cos\epsilon\tau)\\
&- p(1-p)\frac{\lambda^2(\sigma_--\sigma_+)}{|\mu|^2}e^{-(\sigma_--\sigma_+)\tau/2}\sin\epsilon\tau.
\end{aligned} \tag{4.27}$$

Notice again that for $\sigma_-\to+0$ one obtains from (4.27) the one-step energy variation for the ideal cavity (4.12).

Summarising (4.23) and (4.27), we obtain the energy increment (4.25) which is due to impact of the open cavity effects (4.23) and to the atomic beam pumping (4.27):

$$\Delta\mathcal{E}_\sigma(n\tau-0,(n-1)\tau-0)= \tag{4.28}$$
$$=\epsilon\,(\frac{\sigma_+}{\sigma_- -\sigma_+}-N_\sigma(0))(1-e^{-(\sigma_- -\sigma_+)\tau})e^{-(n-1)(\sigma_- -\sigma_+)\tau}$$
$$+\,p(1-p)\frac{2\lambda^2\epsilon}{|\mu|^2}(1-e^{-(\sigma_- -\sigma_+)\tau/2}\cos\epsilon\tau)e^{-(n-1)(\sigma_- -\sigma_+)\tau}$$
$$+\,p^2\frac{\lambda^2(\sigma_- -\sigma_+)}{|\mu|^2}\left[e^{-n(\sigma_- -\sigma_+)\tau/2}\sin n\epsilon\tau-e^{-(n-1)(\sigma_- -\sigma_+)\tau/2}\sin(n-1)\epsilon\tau\right].$$

Theorem 4.8. *By virtue of (4.28) the total energy variation between initial state at the moment $t_0:=-0$, when the cavity is empty, and the moment $t_n:=n\tau-0$, just before the n-th atom is ready to leave the cavity, is*

$$\Delta\mathcal{E}_\sigma(t_n,t_0)=\sum_{k=1}^{n}\Delta\mathcal{E}_\sigma(k\tau-0,(k-1)\tau-0) \tag{4.29}$$
$$=\epsilon\,(\frac{\sigma_+}{\sigma_- -\sigma_+}-N_\sigma(0))(1-e^{-n(\sigma_- -\sigma_+)\tau})$$
$$+\,p(1-p)\frac{2\lambda^2\epsilon}{|\mu|^2}(1-e^{-(\sigma_- -\sigma_+)\tau/2}\cos\epsilon\tau)\frac{1-e^{-n(\sigma_- -\sigma_+)\tau}}{1-e^{-(\sigma_- -\sigma_+)\tau}}$$
$$+\,p^2\frac{\lambda^2(\sigma_- -\sigma_+)}{|\mu|^2}e^{-n(\sigma_- -\sigma_+)\tau/2}\sin n\epsilon\tau\ .$$

Here $N_\sigma(0)=\omega^{t_0}_{\mathcal{S},\sigma}(b^*b\otimes\mathbb{1})$ is the initial number of photons in the cavity.

Remark 4.9. Note that the total energy variation (4.29) is due to evolution of the photon number in the open cavity (3.39) and the variation of the interaction energy (4.21), that give

$$\Delta\mathcal{E}_\sigma(t_n,t_0)=\epsilon\omega^{n\tau}_{\mathcal{S},\sigma}(b^*b\otimes\mathbb{1})+\lambda\,\omega^{n\tau}_{\mathcal{S},\sigma}((b^*+b)\otimes\eta_n) \tag{4.30}$$
$$-\,\epsilon\omega^{t_0}_{\mathcal{S},\sigma}(b^*b\otimes\mathbb{1})\ .$$

For $\sigma_- -\sigma_+>0$ it is uniformly bounded from above

$$\Delta\mathcal{E}_\sigma(t_n,t_0)\le\epsilon\,\frac{\sigma_+}{\sigma_- -\sigma_+}+\frac{2\lambda^2\epsilon}{|\mu|^2}\frac{p(1-p)}{1-e^{-(\sigma_- -\sigma_+)\tau/2}} \tag{4.31}$$
$$+\,p^2\frac{\lambda^2(\sigma_- -\sigma_+)}{|\mu|^2}.$$

The lower bound of (4.29) is also evident. It strongly depends on the initial condition $N_\sigma(0)$ and can be negative.

The long-time asymptotic of (4.29), or (4.30), is

$$\Delta\mathcal{E}_\sigma:=\lim_{n\to\infty}\Delta\mathcal{E}_\sigma(t_n,t_0)=\epsilon\,(\frac{\sigma_+}{\sigma_- -\sigma_+}-N_\sigma(0)) \tag{4.32}$$
$$+\,p(1-p)\frac{2\lambda^2\epsilon}{|\mu|^2}\frac{1-e^{-(\sigma_- -\sigma_+)\tau/2}\cos\epsilon\tau}{1-e^{-(\sigma_- -\sigma_+)\tau}}\ .$$

From (4.32) one gets that in the open cavity with $\sigma_- - \sigma_+ > 0$ the asymptotic of the total-energy variation is bounded from above and from below

$$\Delta\mathcal{E}_\sigma \leq \epsilon\,(\frac{\sigma_+}{\sigma_- - \sigma_+} - N_\sigma(0)) + \frac{2\lambda^2\epsilon}{|\mu|^2}\frac{p(1-p)}{1-e^{-(\sigma_- - \sigma_+)\tau/2}}\ , \tag{4.33}$$

$$\Delta\mathcal{E}_\sigma \geq \epsilon\,(\frac{\sigma_+}{\sigma_- - \sigma_+} - N_\sigma(0)) + \frac{2\lambda^2\epsilon}{|\mu|^2}\frac{p(1-p)}{1+e^{-(\sigma_- - \sigma_+)\tau/2}}\ . \tag{4.34}$$

For the short-time regime $n\tau \ll 1$ one gets for (4.29)

$$\begin{aligned}\Delta\mathcal{E}_\sigma(t_n, t_0) &= n\tau\epsilon\,\sigma_+ - n\tau(\sigma_- - \sigma_+)N_\sigma(0) \\ &+ n\tau p(1-p)\frac{2\lambda^2\epsilon}{|\mu|^2}(1-\cos\epsilon\tau)\frac{\sigma_- - \sigma_+}{1-e^{-(\sigma_- - \sigma_+)\tau}} \\ &+ n\tau p^2\frac{\lambda^2(\sigma_- - \sigma_+)\epsilon}{|\mu|^2} + \mathcal{O}((n\tau)^2)\ ,\end{aligned} \tag{4.35}$$

i.e. a linear asymptotic behaviour.

Another asymptotics one finds for the small difference between leaking and external pumping: $\sigma_- - \sigma_+ \to 0$. Then (4.29) yields linear behaviour

$$\Delta\mathcal{E}_\sigma(t_n, t_0) = n\tau\epsilon\,\sigma_+ + np(1-p)\frac{2\lambda^2}{\epsilon}(1-\cos\epsilon\tau)\ , \tag{4.36}$$

which is a *growing* of the total energy due to the both external and atomic beam pumping. Note that in this limit the energy variation $\Delta\mathcal{E}_\sigma(t_n, t_0)$ is not bounded from above (4.31), (4.33).

This case coincides with result for the ideal cavity (Theorem 4.2) when the rate of environmental pumping $\sigma_+ = 0$.

4.3. Entropy production in the ideal cavity.

One of the central quantities to study in the non-equilibrium statistical mechanics is the entropy production (or the entropy production rate). We refer to the series of papers [BJM1]-[BJM3] by Bruneau, Joye, and Merkli, for a detailed discussion of this quantity in the context of open quantum systems with repeated interactions, and we shall adopt definitions of these authors.

Let ρ and ρ_0 be two normal states on the algebra $\mathfrak{A}(\mathcal{H})$. We define the *relative entropy* $\mathrm{Ent}(\rho|\rho_0)$ of the state ρ with respect to a "reference" state ρ_0 by

$$\mathrm{Ent}(\rho|\rho_0) := \mathrm{Tr}(\rho\ln\rho - \rho\ln\rho_0) \geq 0\,. \tag{4.37}$$

The non-negativity follows from the Jensen inequality: $\mathrm{Tr}(\rho\ln B) \leq \ln\mathrm{Tr}(\rho B)$, applied to observable $B := \rho_0/\rho$.

Here we calculate the entropy production:

$$\Delta S(t) := \mathrm{Ent}(\rho_{\mathcal{S}}(t)|\rho_{\mathcal{S}}^{ref}) - \mathrm{Ent}(\rho_{\mathcal{S}}(0)|\rho_{\mathcal{S}}^{ref})\ , \tag{4.38}$$

for the ideal cavity: $\sigma_- = \sigma_+ = 0$, with dynamics $\rho_{\mathcal{S}} : t \mapsto \rho_{\mathcal{S}}(t)$ (1.16), and for a reference state $\rho_{\mathcal{C}}^{ref} \otimes \rho_{\mathcal{A}}$. To make a contact with thermodynamics, we suppose that all atoms of the beam are in the *Gibbs state* with the temperature $1/\beta$. Formally this can be written as

$$\rho_{\mathcal{A}}(\beta) := \bigotimes_{n\geq 1} \rho_{\mathcal{A}_n}(\beta) \ , \quad \rho_{\mathcal{A}_n}(\beta) := \frac{e^{-\beta H_{\mathcal{A}_n}}}{Z(\beta)} \ , \tag{4.39}$$

see (1.5). Since

$$\rho_{\mathcal{S}}^{ref} = \rho_{\mathcal{C}}^{ref} \otimes \rho_{\mathcal{A}} = (\rho_{\mathcal{C}}^{ref} \otimes \mathbb{1})(\mathbb{1} \otimes \rho_{\mathcal{A}}) \ , \tag{4.40}$$

and since for the unitary dynamics (1.9)

$$\mathrm{Tr}\{\rho_{\mathcal{S}}(t) \ln \rho_{\mathcal{S}}(t)\} = \mathrm{Tr}\{\rho_{\mathcal{S}}(0) \ln \rho_{\mathcal{S}}(0)\}, \tag{4.41}$$

the relative entropy (4.38) is

$$\begin{aligned} \Delta S(t) :&= \mathrm{Tr}_{\mathcal{C}}\{[\rho_{\mathcal{C}}^{(0)} - \rho_{\mathcal{C}}^{(n)}] \ln \rho_{\mathcal{C}}^{ref}\} \\ &- \beta \sum_{k=1}^{n} \mathrm{Tr}\{[\rho_{\mathcal{S}}(0) - \rho_{\mathcal{S}}(n\tau)](\mathbb{1} \otimes H_{\mathcal{A}_k})\} \ , \end{aligned} \tag{4.42}$$

for $t = n\tau + \nu$, see (1.15). Here $\rho_{\mathcal{C}}^{(n)}$ is defined by (1.22).

Remark 4.10. For any Hamiltonian H_n that acts non-trivially on $\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{A}_n}$ and for any Hamiltonian $H_{\mathcal{A}_k}$ acting on $\mathcal{H}_{\mathcal{A}_k}$ one gets $[H_n, H_{\mathcal{A}_k}] = 0$ if $n \neq k$. Note that in our model (1.21) implies $[H_n, H_{\mathcal{A}_k}] = 0$ for any n, k.

Therefore, by virtue of (1.22) and (1.23) we obtain

$$\begin{aligned} &\mathrm{Tr}\{\rho_{\mathcal{S}}(n\tau) \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \\ &= \mathrm{Tr}\{e^{-i\tau H_n} ... e^{-i\tau H_{k+1}} \rho_{\mathcal{S}}(k\tau) e^{i\tau H_{k+1}} \ldots e^{i\tau H_n} \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \\ &= \mathrm{Tr}\{\rho_{\mathcal{S}}((k\tau) \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \ . \end{aligned} \tag{4.43}$$

Then by the same arguments one gets also that

$$\begin{aligned} &\mathrm{Tr}\{\rho_{\mathcal{S}}(0) \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \\ &= \mathrm{Tr}\{e^{-i\tau H_{k-1}} ... e^{-i\tau H_1} \rho_{\mathcal{S}}(0) e^{i\tau H_1} \ldots e^{i\tau H_{k-1}} \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \\ &= \mathrm{Tr}\{\rho_{\mathcal{S}}((k-1)\tau) \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \ . \end{aligned} \tag{4.44}$$

In the case of our model (see Remark 4.10) $H_{\mathcal{A}_k} = e^{i\tau H_k}(H_{\mathcal{A}_k})e^{-i\tau H_k}$. Therefore the last formula gets the form

$$\mathrm{Tr}\{\rho_{\mathcal{S}}(0) \ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} = \mathrm{Tr}\{\rho_{\mathcal{S}}(k\tau) \ (\mathbb{1} \otimes H_{\mathcal{A}_k}). \tag{4.45}$$

Equations (4.43) and (4.45) shows that the second term in the entropy production (4.42) *vanishes*.

If we suppose that the reference state is the Gibbs state (1.29) at temperature $1/\beta_{\mathcal{C}}$, then by (1.26) one gets

$$\begin{aligned} \Delta S(t) &= \mathrm{Tr}_{\mathcal{C}}\{[\rho_{\mathcal{C}}^{(0)} - \rho_{\mathcal{C}}^{(n)}] \ln \rho_{\mathcal{C}}^{ref}\} \\ &= \mathrm{Tr}_{\mathcal{C}}\{[\rho_{\mathcal{C}}^{(0)} - \rho_{\mathcal{C}}^{(n)}](-\beta_{\mathcal{C}} \, \epsilon \, b^* b)\} \\ &= \beta_{\mathcal{C}} \, \epsilon (N(t) - N(0)) \ , \end{aligned} \tag{4.46}$$

where $N(t)$ is the mean photon number defined by (1.26).

Let us define by $\Delta\mathcal{E}^{\mathcal{C}}(t) = \epsilon(N(t) - N(0))$ the energy variation of the cavity due to the photon number evolution. Then (4.46) expresses the 2nd Law of Thermodynamics

$$\Delta S(t) = \beta_{\mathcal{C}}\, \Delta\mathcal{E}^{\mathcal{C}}(t) \ , \tag{4.47}$$

for the pumping by atomic beam. Note that relation (4.47) does not depend on the initial cavity state $\rho_{\mathcal{C}}^{(0)}$.

Remark 4.11. In general, when $[H_n, H_{\mathcal{A}_n}] \neq 0$, the combination of (4.42) with (4.43) and (4.44) yield for the entropy production at the moment $t = n\tau + \nu$ the expression:

$$\begin{aligned}\Delta S(t) :&= \mathrm{Tr}_{\mathcal{C}}\{[\rho_{\mathcal{C}}^{(0)} - \rho_{\mathcal{C}}^{(n)}] \ln \rho_{\mathcal{C}}^{ref}\} \\ &\quad + \beta \sum_{k=1}^{n} \mathrm{Tr}\{[\rho_{\mathcal{S}}(k\tau) - \rho_{\mathcal{S}}((k-1)\tau)]\ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \ .\end{aligned} \tag{4.48}$$

The last term in (4.48) can be rewritten into the standard form [BJM1]-[BJM3], if one uses the identities:

$$\begin{aligned}&\mathrm{Tr}\{\rho_{\mathcal{S}}(k\tau)(\mathbb{1} \otimes H_{\mathcal{A}_k})\} = \mathrm{Tr}\{e^{\tau L_k} \ldots e^{\tau L_1}(\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}})\ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \\ &= \mathrm{Tr}\{e^{-i\tau H_k}(e^{\tau L_{k-1}} \ldots e^{\tau L_1}[\rho_{\mathcal{C}} \otimes \bigotimes_{n=1}^{k-1} \rho_n]) \otimes \rho_k e^{i\tau H_k}\ (\mathbb{1} \otimes H_{\mathcal{A}_k})\}[\mathbb{1} \otimes \bigotimes_{m>k} \rho_m] \\ &= \mathrm{Tr}\{(\rho_{\mathcal{C}}^{(k-1)} \otimes \rho_k) e^{i\tau H_k}\ (\mathbb{1} \otimes H_{\mathcal{A}_k}) e^{-i\tau H_k}\} \ ,\end{aligned}$$

and

$$\mathrm{Tr}\{\rho_{\mathcal{S}}((k-1)\tau)\ (\mathbb{1} \otimes H_{\mathcal{A}_k})\} = \mathrm{Tr}\{\rho_{\mathcal{C}}^{(k-1)} \otimes \rho_k H_{\mathcal{A}_k}\} \ .$$

For $t = n\tau + \nu$ this gives the formula for the entropy production in the ideal cavity:

$$\begin{aligned}&\Delta S(t) = \mathrm{Tr}_{\mathcal{C}}\{[\rho_{\mathcal{C}}^{(0)} - \rho_{\mathcal{C}}^{(n)}] \ln \rho_{\mathcal{C}}^{ref}\} \\ &+ \beta \sum_{k=1}^{n} \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{A}_k}}\{(\rho_{\mathcal{C}}^{(k-1)} \otimes \rho_k)[e^{i\tau H_k}(\mathbb{1} \otimes H_{\mathcal{A}_k}) e^{-i\tau H_k} - \mathbb{1} \otimes H_{\mathcal{A}_k}]\} \ .\end{aligned} \tag{4.49}$$

One expects that for an open cavity the gain and loss of photons would result in a corresponding additional flux of entropy due to the quantum Markov evolution. It is not yet completely clear (see e.g.[FaRe]) how to define this entropy production correctly and whether contributions of these two processes are independent. Therefore, we leave the analysis of the entropy flux for the open cavity to be considered in the future.

5. Concluding Remarks

From Theorem 1.2 we learn that in the ideal cavity the photon number increases linearly in time up to bounded oscillations, (see Figure 1, green curve) The rate of the growth is non-zero for any value of the probability of excited atoms in the beam except $p = 0$ and $p = 1$. For large time it is independent

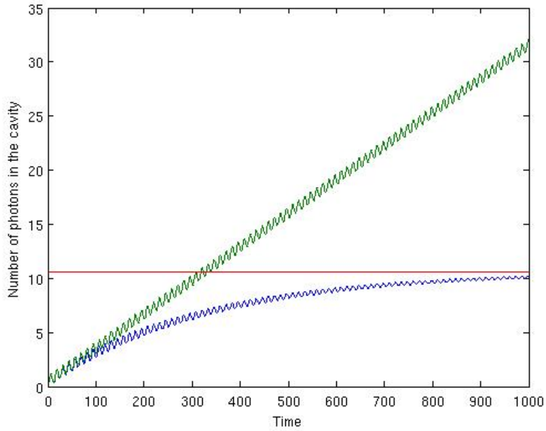

FIGURE 1. In blue, we have plotted the photon number in an open cavity with parameters $\sigma_- = 0.003$, $\sigma_+ = 0$, $\epsilon = 0.5$, $\tau = 0.5$, $p = \frac{1}{2}$, $\frac{\lambda^2}{|\mu|^2} = 1$ and vanishing initial number of photons $N_\sigma(0) = 0$. The green curve is the mean-value of photons in the ideal cavity for the same parameters and initial condition $N(0) = 0$, except $\sigma_- = 0$. The expressions for these quantities are given by (1.28) for the ideal cavity and by (3.39) for the open cavity. The red line is the asymptotic value (1.58) of the mean photon number in the open cavity.

of the initial state $\rho_\mathcal{C}$. The rate of the linear growth of the mean-value of photons $N(t)$ with respect to the time $t = n\tau$ can be seen from (1.28) since

$$\frac{N(t)}{n\tau} = \frac{N(0)}{n\tau} + p(1-p)\,\frac{2\lambda^2}{\tau\epsilon^2}\,(1 - \cos\epsilon\tau) + \frac{p^2}{n\tau}\,\frac{2\lambda^2}{\epsilon^2}(1 - \cos n\epsilon\tau)\ . \quad (5.1)$$

Hence, for $N(0) = 0$ one gets linear growth modulo bounded oscillations:

$$\frac{N(t)}{n\tau} = p(1-p)\,\frac{2\lambda^2}{\tau\epsilon^2}\,(1 - \cos\epsilon\tau) + \frac{p^2}{n\tau}\,\frac{2\lambda^2}{\epsilon^2}(1 - \cos n\epsilon\tau)\ . \quad (5.2)$$

This linear growth of the photon number is observed in experiments with one-atom masers for the high-quality resonators (nearly ideal cavities) [MWM].

As can be clearly seen in Figure 1 (blue curve), the mean photon number $N_\sigma(t)$ in the open cavity (3.39) is also initially increasing linearly up to bounded oscillations, but then it stabilizes for large times. Similarly to (5.1) the rate of the growth in the open cavity with respect to the time $t = n\tau$ can be deduced from

$$\begin{aligned}\frac{N_\sigma(n\tau)}{n\tau} &= e^{-n(\sigma_- - \sigma_+)\tau}\frac{N_\sigma(0)}{n\tau} \\ &+ p(1-p)\frac{2\lambda^2}{|\mu|^2}(1 - e^{-(\sigma_- - \sigma_+)\tau/2}\cos\epsilon\tau)\frac{1 - e^{-n(\sigma_- - \sigma_+)\tau}}{n\tau(1 - e^{-(\sigma_- - \sigma_+)\tau})} \\ &+ \frac{p^2}{n\tau}\frac{2\lambda^2}{|\mu|^2}(1 - e^{-n(\sigma_- - \sigma_+)\tau/2}\cos n\epsilon\tau) + \frac{\sigma_+}{\sigma_- - \sigma_+}\frac{1 - e^{-n(\sigma_- - \sigma_+)\tau}}{n\tau}\ .\end{aligned} \tag{5.3}$$

which for $\sigma_+ = 0$ and then $\sigma_- = 0$ gives (5.1).

Let $N_\sigma(0) = 0$ and $\sigma_+ = 0$. Since $\mu = (\sigma_- - \sigma_+)/2 + i\epsilon$, the short-time behaviour of (5.3) for $n(\sigma_- - \sigma_+)\tau \ll 1$ is linear (modulo bounded oscillations):

$$\begin{aligned}\frac{N_\sigma(n\tau)}{n\tau} &= p(1-p)\frac{2\lambda^2}{\tau\epsilon^2}(1 - \cos\epsilon\tau) + \frac{p^2}{n\tau}\frac{2\lambda^2}{\epsilon^2}(1 - \cos n\epsilon\tau) + \\ &+ \mathcal{O}((n(\sigma_- - \sigma_+)\tau)^2)\ ,\end{aligned} \tag{5.4}$$

and asymptotically it is close to (5.2). This is clearly visible in the Figure 1 for $n < 100$.

There are several generalisations of the beam-cavity problem considered in this paper that could be handled by suitable modifications of our methods. First, there is the detuned case, when the distance between atoms is greater than the length of the cavity, $d > l$. This is a situation where there is still at most one atom interacting with the cavity at any given time, but there are time intervals without an atom present. The modifications needed to analyze this situation are straightforward. A more interesting generalization would be to consider random interatomic distances $d \geq l$, when again still there is at most one atom in the cavity, but they arrive randomly.

Due to properties of the apparatus that produces the atom beam, one may expect short range correlations in the chain of atoms. Such a correlated atomic beam can be described by a classical Markov chain or, more generally, by a so-called Finitely Correlated State [FNW]. Calculating the asymptotic behavior of the cavity in this situation will be a bit more complicated but should still be doable.

We would like also to mention the following two other problems. The first is to consider a cavity interaction with 'soft atoms'. One of the possible interactions is of the form $b^* \otimes \sigma^- + b \otimes \sigma^+$, i.e., interaction of the Jaynes-Cummings type. Here in contrast to (1.3) the atomic operators $\sigma^\pm := (\sigma^x \pm i\sigma^y)/2$ are *off-diagonal*, constructed with the Pauli matrices σ^x and σ^y. For $p < 1/2$ and for an ideal cavity this problem has been studied in [BPi]. It was shown that the Jaynes-Cummings interaction allows the cavity state to converge to some explicit thermal state. So, the pumping is saturated and the photon number expectation is well-defined and bounded.

The second problem is to calculate the entropy production for the Kossakowski-Lindblad open cavity and to determine its relation to the energy flux in the frame work of our model. For a discussion of the entropy production problem for the quantum Markov semigroups see, e.g., the recent paper [FaRe].

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Appendix: Open One-Mode Photon Cavity

Here we collect some remarks and recall certain statements concerning the quantum theory of open systems, see e.g. [Dav1, AlFa, AJPII]. To this end we consider example of the open one-mode photon cavity $\mathcal{C}$. This soluble example is useful for our discussion of the leaky cavity pumped by atomic beam starting on Section 3.

A.1 Markovian Master Equation. We treat the cavity: $H_{\mathcal{C}} = \epsilon\, b^* b$, interacting with external reservoir $\mathcal{R}$ in the framework of the Markovian approach [AJPII, AJPIII]. Then $\mathcal{R}$ is a source of a *leaking*, which decreases the cavity energy with the rate σ_-, or/and of a *pumping* with the rate σ_+. The corresponding Markovian master equation for evolution of the normal cavity states $\omega_{\mathcal{C}}^t(\cdot)$ with trace-class density matrices $\rho_{\mathcal{C}}(t) \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ is extension of the Hamiltonian dynamics by Kossakowski-Lindblad damping (leaking) and pumping terms [Al, AlFa]:

$$\begin{aligned}\frac{d}{dt}\rho_{\mathcal{C}}(t) = L_{\mathcal{C},\sigma}(\rho_{\mathcal{C}}(t)) &:= -i[H_{\mathcal{C}}, \rho_{\mathcal{C}}(t)] \qquad (5.5)\\ &+ \frac{1}{2}\sigma_-([b\rho_{\mathcal{C}}(t), b^*] + [b, \rho_{\mathcal{C}}(t)b^*]) + \frac{1}{2}\sigma_+([b^*\rho_{\mathcal{C}}(t), b] + [b^*, \rho_{\mathcal{C}}(t)b])\ .\end{aligned}$$

Here $\rho_{\mathcal{C}}(t) \in \mathrm{dom}(L_{\mathcal{C},\sigma})$ and parameters $\sigma_-, \sigma_+ \geq 0$.

Note that evolution (5.5) for $\rho_{\mathcal{C}} := \rho_{\mathcal{C}}(0) \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$:

$$\mathcal{L}_{\mathcal{C},\sigma}^t : \rho_{\mathcal{C}} \mapsto \rho_{\mathcal{C}}(t)\ , \qquad (5.6)$$

is the case of *trace-norm* continuous semigroup: $\mathcal{L}_{\mathcal{C},\sigma}^t := e^{t\, L_{\mathcal{C},\sigma}}$ on the Banach space $\mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$, with *unbounded* generator $L_{\mathcal{C},\sigma}$, see e.g. [Za], Ch.2.4.

It is known that this case of quantum Markovian dynamics (5.6) needs a special care, see e.g. [Dav1], [Si, ChFa], but for the open one-mode photon cavity $\mathcal{C}$ all necessary properties can be checked explicitly.

Denote by $\widehat{L}_{\mathcal{C},\sigma}$ the non-Hamiltonian part of unbounded generator (5.5). Then for any $\rho \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ one has:

$$\widehat{L}_{\mathcal{C},\sigma}(\rho) = \sum_{\alpha=\downarrow,\uparrow} \sigma_\alpha\{V_\alpha \rho V_\alpha^* - \frac{1}{2}(V_\alpha^* V_\alpha \rho + \rho V_\alpha^* V_\alpha)\}\ ,\ V_\downarrow = b\ ,\ V_\uparrow = b^*\ . \qquad (5.7)$$

By virtue of the trace cyclicity this canonical form of the Kossakowski-Lindblad generator:

$$L_{\mathcal{C},\sigma} := -i\,[H_{\mathcal{C}}, \cdot] + \widehat{L}_{\mathcal{C},\sigma}\ \ , \qquad (5.8)$$

ensures the *trace-preserving* property of dynamics (5.5):

$$\frac{d}{dt}\mathrm{Tr}_{\mathcal{C}}\rho_{\mathcal{C}}(t) = 0. \qquad (5.9)$$

Note that by virtue of (5.5) the Markovian dynamics (5.6) is also unity-preserving: $\mathcal{L}_{\mathcal{C},\sigma}^t(\mathbb{1}) = \mathbb{1}$ for $t \geq 0$.

To check another important property: the (complete) *positivity* of the trace-norm continuous semigroup $\{\mathcal{L}_{\mathcal{C},\sigma}^t\}_{t\geq 0}$ on the space $\mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$, let us present its generator (5.8) as $L_{\mathcal{C},\sigma} := \Phi - \Gamma$, where

$$\Phi(\rho) := \sum_{\alpha=\downarrow,\uparrow} \sigma_\alpha \, V_\alpha \, \rho \, V_\alpha^* \quad , \qquad \sigma_\alpha \geq 0 \quad , \tag{5.10}$$

$$\Gamma(\rho) := \Psi \, \rho + \rho \, \Psi^* \quad \text{with} \quad \Psi := i \, H_{\mathcal{C}} + \frac{1}{2} \sum_{\alpha=\downarrow,\uparrow} \sigma_\alpha \, V_\alpha \, V_\alpha^* \ . \tag{5.11}$$

First we reduce our analysis to positivity and we postpone the question concerning complete positivity to the end of this section and to the Heisenberg picture of quantum dynamics (5.6), see subsection A.2.

To see that dynamical semigroup $\{e^{t\,\Phi}\}_{t\geq 0}$ with generator (5.10) enjoy the property of the positivity, notice that by (5.10) one gets for the trace-continuous maps $\rho \mapsto \rho_\Phi(t) := e^{t\,\Phi}(\rho)$:

$$\frac{d}{dt}\rho_\Phi(t) = \Phi(\rho_\Phi(t)) = \sum_{\alpha=\downarrow,\uparrow} \sigma_\alpha \, V_\alpha \, \rho_\Phi(t) \, V_\alpha^* \ . \tag{5.12}$$

Let $\rho \in \mathrm{dom}(\Phi) \subset \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ and $\rho \geq 0$. Then (5.10) implies that $\Phi(\rho) \geq 0$ and that equation (5.12) is positivity-preserving. This yields positivity of the solution $\rho_\Phi(t)$ for $t \geq 0$, if $\rho_\Phi(t=0) = \rho$.

To see that semigroup $\{e^{-t\,\Gamma}\}_{t\geq 0}$ is also a family of positive mapping on $\mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ note that (5.11) yields

$$\frac{d}{dt}\rho_\Gamma(t) = -\Gamma(\rho_\Gamma(t)) = -(\Psi \, \rho_\Gamma(t) + \rho_\Gamma(t) \, \Psi^*) = \frac{d}{dt}\left(e^{-t\,\Psi} \, \rho \, e^{-t\,\Psi^*}\right) \ . \tag{5.13}$$

Then the mapping $e^{-t\,\Gamma} : \rho \mapsto e^{-t\,\Psi} \rho \, e^{-t\,\Psi^*}$ is positive. For $\rho \in \mathfrak{C}_1(\mathcal{H}_{\mathcal{C}})$ we denote by $\rho_\Gamma(t) := e^{-t\,\Gamma}(\rho)$ the solution of (5.13). This operator is positive for $\rho \geq 0$.

By virtue of (5.12) and (5.13) the composition of two maps: $F(t) : \rho \mapsto e^{t\,\Phi}(e^{-t\,\Gamma}(\rho))$, is a positive trace-norm continuous mapping. Note that the powers of $F(t)$ are also positive maps. Then it is also true for dynamical semigroup with generator (5.8), since by the Trotter product formula one gets

$$\mathcal{L}_{\mathcal{C},\sigma}^t = e^{t\,L_{\mathcal{C},\sigma}} = \|\cdot\|_1 - \lim_{n\to\infty} \left(e^{t\,\Phi/n} \; e^{-t\,\Gamma/n}\right)^n \ , \tag{5.14}$$

in the trace-norm topology [Za]. This remark ensures, in particular, the $\mathfrak{C}_1$-continuity of the limit (5.14). Note also that by (5.5) and (5.6) the limit (5.14) is unity-preserving (or Markov) semigroup.

Recall that essential in the concept of open systems is a coupling of some small sub-system with "environment", which is a certain large (even infinite) system. The mathematical description of this concept involves a tensor product of corresponding configurations spaces and states or algebras of observables. Then the positivity-preserving evolution due to the master equation for a coupled system, must be robust for forming the tensor products [Dav1]. Since tensor product of two positive maps might fail to be positive ([AlFa], Ch.8.3), the evolution (5.6) has to verify a stronger condition than positivity established in (5.14).

We recall now the complete positivity property and constrains that it implies on dynamical semigroup (5.6) and on its generator [Dav1, AlFa, AJPII].

Let $T : \mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)}$ be a positive linear map between two C^*-algebras, i.e. $T(A) \geq 0$, where $A \in \mathfrak{A}^{(1)}$ and $A \geq 0$. For $k = 1, 2$ and for $n \in \mathbb{N}$, let

$$\mathcal{M}_n(\mathfrak{A}^{(k)}) \simeq \mathfrak{A}^{(k)} \otimes \mathcal{M}(\mathbb{C}^n) \ , \tag{5.15}$$

be algebra of $n \times n$ matrices with entries in $\mathfrak{A}^{(k)}$. Each of $\mathcal{M}_n(\mathfrak{A}^{(k)})$ is also a C^*-algebra. If we denote by Id_n the identity matrix from $\mathcal{M}(\mathbb{C}^n)$, then

$$T_n := T \otimes \mathrm{Id}_n : \mathcal{M}_n(\mathfrak{A}^{(1)}) \to \mathcal{M}_n(\mathfrak{A}^{(2)}) \ , \tag{5.16}$$

defines a linear map by acting with T on each of the matrix element of the operator-valued matrix $A_n \in \mathcal{M}_n(\mathfrak{A}^{(1)})$. The positive map T is called n-positive (respectively completely positive), if operator T_n is positive (respectively (5.16) are positive for all $n \geq 1$). For $n = 1$ it obviously reduces to the positive map.

Example 5.1. A simple example shows that this property is quite non-trivial. (For more of them we refer to [AlFa].) Let $\mathfrak{A}^{(k=1,2)} = \mathcal{M}(\mathbb{C}^2)$ be C^*-algebra of square complex matrices. Then the map of matrices $\mathcal{M}(\mathbb{C}^2)$ to the *adjoint*, $T_{adj} : A \to A^*$ is obviously positive. Denote by $\{E_{ij} \in \mathcal{M}(\mathbb{C}^2)\}_{i,j=1,2}$ the set of matrices with 1 in the ij-th entry and zeros elsewhere, i.e. $\mathrm{Id}_2 = \sum_{i,j=1,2} E_{ij}$. To verify whether the map T_{adj} is 2-positive we consider the algebra (5.15):

$$\mathcal{M}_2(\mathcal{M}(\mathbb{C}^2)) \simeq \mathcal{M}(\mathbb{C}^2) \otimes \mathcal{M}(\mathbb{C}^2) \simeq \mathcal{M}(\mathbb{C}^4) \ , \tag{5.17}$$

and the element $E := \sum_{i,j=1,2} E_{ij} \bigotimes E_{ij} \in \mathcal{M}(\mathbb{C}^4)$. Since $E = E^*$ and $E^2 = 2E$, it is positive $E \geq 0$. On the other hand by definition (5.16) one gets

$$T_{adj,2} := T_{adj} \otimes \mathrm{Id}_2 : E \to \begin{bmatrix} T_{adj}(E_{11}) & T_{adj}(E_{12}) \\ T_{adj}(E_{21}) & T_{adj}(E_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5.18}$$

The matrix $T_{adj,2}(E)$ in (5.18) is *not* positive, since its spectrum contains (-1). Therefore, the map T_{adj} is *not* 2-positive either.

One of the important corollary imposed by demand that quantum Markovian dynamics on the Banach space $\mathfrak{C}_1(\mathcal{H})$:

$$\mathcal{L}^t = e^{t\,L} : \rho \mapsto \rho(t) \ , \quad \rho(0) = \rho \in \mathfrak{C}_1(\mathcal{H}) \ , \tag{5.19}$$

must be *completely* positive, is certain restrictions on the form of the generator L, cf. (5.6) and (5.8). For the case of bounded generator, when semigroup (5.19) is continuous in the operator-norm topology of mappings on $\mathfrak{C}_1(\mathcal{H})$, this is just a celebrated Kossakowski-Lindblad result saying that the most general form of L is

$$L := -i\,[H, \cdot] + \widehat{L} \ , \tag{5.20}$$

where non-Hamiltonian part can be presented as

$$\widehat{L}(\rho) := \frac{1}{2}\sum_{\alpha}([\widehat{V}_\alpha \rho\, ,\, \widehat{V}_\alpha^*] + [\widehat{V}_\alpha\, ,\rho\, \widehat{V}_\alpha^*]) \tag{5.21}$$
$$= \sum_{\alpha}\{\widehat{V}_\alpha \rho \widehat{V}_\alpha^* - \frac{1}{2}(\widehat{V}_\alpha^* \widehat{V}_\alpha\, \rho + \rho\, \widehat{V}_\alpha^* \widehat{V}_\alpha)\}\ ,$$

see, e.g., [AlFa], Ch.8. Note that the choice of bounded operators $H = H^*$ and $\{\widehat{V}_\alpha\}_\alpha$ in representation (5.21) is not unique.

The contact between (5.20), (5.21) and representations (5.7),(5.8) as well as with (5.10), (5.11) related to semigroups $\{e^{t\,\Phi}\}_{t\geq 0}$, $\{e^{-t\,\Gamma}\}_{t\geq 0}$ follows through verbatim. Then taking into account that the Stinespring theorem ([AlFa], Ch.8) implies the complete positivity of these two semigroups and using the Trotter product formula one can check the complete positivity of (5.19).

The present results for unbounded H and/or $\widehat{L}$ are more poor. Certain classes of strongly continuous on $\mathfrak{C}_1(\mathcal{H})$, contraction semigroups (5.19) have been studied in [Dav1] and [EvLe1, EvLe2] under condition that operator $\sum_\alpha \widehat{V}_\alpha^* \widehat{V}_\alpha$ is generator of a strongly continuous contraction semigroup on $\mathcal{H}$. For further developments see, e.g., [Dav2, Si, Hol, ChFa].

The Kossakowski-Lindblad representation (5.20), (5.21) explains our choice of the right-hand side in the Markovian Master Equation (5.5), but appeals for a concrete verification of the complete positivity of quantum Markovian dynamics (5.6). Similar to other known cases of the Kossakowski-Lindblad type generators [FrVe], this property of $\mathcal{L}^t_{\mathcal{C},\sigma}$ follows directly from explicit calculations.

A.2 Dual Dynamical Map (Heisenberg Picture). The equivalent (and often more convenient) is the abstract version of the reduced Markovian dynamics A.1 on the algebra of observables $\mathfrak{A}(\mathcal{H}_\mathcal{C})$, i.e. the quantum dynamical semigroup $\mathcal{L}^{t\,*}_{\mathcal{C},\sigma} := (\mathcal{L}^t_{\mathcal{C},\sigma})^*$ in the *dual* Heisenberg picture [AlFa, AJPI].

Recall that in a general setting the C^*-dynamical system is a pair $(\mathfrak{A}, \tau^t)$ with two properties. First, $\mathfrak{A}$ is a unital C^*-algebra, i.e. $\mathbb{1} \in \mathfrak{A}$. Second, τ^t is a strongly continuous one-parameter $*$-automorphism of $\mathfrak{A}$, i.e. for any $A \in \mathfrak{A}$ the map: $t \mapsto \tau^t(A)$, is continuous in the norm topology of $\mathfrak{A}$.

Since quantum states belong to trace-class $\mathfrak{C}_1(\mathcal{H})$, which is not a unital C^*-algebra, this framework is not satisfactory for Markovian dynamics (5.19) on the Banach space $\mathfrak{C}_1(\mathcal{H})$, although (5.19) is a strongly continuous $*$-automorphism even for unbounded generator L.

To define a dynamics, which is dual to the Schrödinger picture (5.19), we recall that semigroup $\{\mathcal{L}^t\}_{t\geq 0}$ serves to calculate evolution of the *normal* states $\{\omega^t(\cdot)\}_{t\geq 0}$ on observables $A \in \mathfrak{B}(\mathcal{H})$:

$$\omega^t(A) = \mathrm{Tr}_\mathcal{H}(\mathcal{L}^t(\rho)\, A) =: \langle \mathcal{L}^t(\rho), A\rangle\ ,\quad \rho \in \mathfrak{C}_1(\mathcal{H})\ . \tag{5.22}$$

Here $\mathfrak{B}(\mathcal{H})$ denote the C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$ and ρ is a density-matrix operator with the trace-norm $\|\rho\|_{\mathfrak{C}_1} = 1$.

Recall that by virtue of (5.22) the Banach space of bounded operators on $\mathcal{H}$ is topologically dual of $\mathfrak{C}_1(\mathcal{H})$: $\mathfrak{B}(\mathcal{H}) = (\mathfrak{C}_1(\mathcal{H}))^*$. This means that the map $A \mapsto \langle \cdot, A\rangle$ is an isometric isomorphism of $\mathfrak{B}(\mathcal{H})$ onto the set of linear continuous functionals $(\mathfrak{C}_1(\mathcal{H}))^*$, defined on the space $\mathfrak{C}_1(\mathcal{H})$. Then semi-norms generated by this duality

$$\{\mathfrak{N}_\rho(A) := |\langle \rho, A\rangle|\}_{\rho\in\mathfrak{C}_1(\mathcal{H})} \ , \quad A \in \mathfrak{B}(\mathcal{H}) \ , \tag{5.23}$$

define on the Banach space $\mathfrak{B}(\mathcal{H})$ the weak*-topology, which coincides with the operator σ-weak topology, and for the operator norm of A one gets:

$$\|A\| = \sup_{\rho\in\mathfrak{C}_1(\mathcal{H})} \frac{|\langle \rho, A\rangle|}{\|\rho\|_{\mathfrak{C}_1}} \ . \tag{5.24}$$

see e.g. [ReSiI] or [AJPI]. Duality (5.22) defines also the *adjoint* semigroup $\{\mathcal{L}^{t\,*}\}_{t\geq 0}$ on the dual space $(\mathfrak{C}_1(\mathcal{H}))^* = \mathfrak{B}(\mathcal{H})$:

$$\langle \mathcal{L}^t(\rho), A\rangle = \langle \rho, \mathcal{L}^{t\,*}(A)\rangle \ , \quad \rho \in \mathfrak{C}_1(\mathcal{H}) \ , \ A \in (\mathfrak{C}_1(\mathcal{H}))^* \ . \tag{5.25}$$

In general, the adjoint semigroup is not strongly continuous on the dual Banach space $(\mathfrak{C}_1(\mathcal{H}))^*$, although (5.25) and the strong continuity of (5.19) trivially imply the weak*-continuity of $\{\mathcal{L}^{t\,*}\}_{t\geq 0}$ on this space.

Even thought $\{\mathcal{L}^{t\,*}\}_{t\geq 0}$ is not necessarily strongly continuous on $\mathfrak{B}(\mathcal{H})$, one can still associate with this semigroup a generator $\widetilde{L}$ in the weak*-topology:

$$\widetilde{L}\,A := w^* - \lim_{t\to+0} \frac{1}{t}(\mathcal{L}^{t\,*}A - A) \ , \tag{5.26}$$

with domain

$$\mathrm{dom}(\widetilde{L}) := \{A \in \mathfrak{B}(\mathcal{H}) : w^* - \lim_{t\to+0} \frac{1}{t}(\mathcal{L}^{t\,*}A - A) \ \ \exists \ \} \ . \tag{5.27}$$

It turns out that generator $\widetilde{L}$ is weak*-densely defined, closed operator, which coincides, $\widetilde{L}A = L^*A$, with the adjoint operator L^*, i.e.

$$\begin{aligned} &\mathrm{dom}(\widetilde{L}) = \mathrm{dom}(L^*) := \\ &\{A \in \mathfrak{B}(\mathcal{H}) : \exists\ A' \in \mathfrak{B}(\mathcal{H}) \ s.t.\ \langle \rho, A'\rangle = \langle L(\rho), A\rangle \text{ for all } \rho \in \mathrm{dom}(L)\} \ . \end{aligned} \tag{5.28}$$

Although the weak*-topology is even *weaker* than the weak topology on the Banach space $\mathfrak{B}(\mathcal{H})$ the above arguments make legitime the characterization of semigroup $\{\mathcal{L}^{t\,*} := e^{t\,L^*}\}_{t\geq 0}$ by a generator in a close similarity to strongly continuous case, see [BrRo1], Ch.3 for further details. The minimal price is that instead of the C^*-algebra of bounded operators $\mathfrak{B}(\mathcal{H})$ one must consider this space endowed by a weaker, namely the weak*-topology.

Recall that *von Neumann algebra* is a C^*-algebra acting on $\mathcal{H}$, containing identity operator and closed in the weak operator topology. It has enough room for the weak*-continuous semigroup $\{e^{t\,L^*}\}_{t\geq 0}$. The Banach space $\mathfrak{B}(\mathcal{H})$ is example of a von Neumann algebra, while the Banach space $\mathfrak{C}_1(\mathcal{H})$ is evidently not.

Let $\mathfrak{M} \subseteq \mathfrak{B}(\mathcal{H})$ be a von Neumann algebra and the weak*-continuous for all $A \in \mathfrak{M}$ map $t \mapsto \tau^t(A)$ be a (semi)group of $*$-automorphisms of $\mathfrak{M}$. Then the pair $(\mathfrak{M}, \tau^t)$ is called a W^*-dynamical system. In our case $\tau^t = \mathcal{L}^{t\,*}$.

If the semigroup (5.19) is trace-preserving (5.9), then the adjoint semigroup (5.25) is a unity-preserving ($\mathcal{L}^{t\,*}(\mathbb{1}) = \mathbb{1}$) contraction: $\|\mathcal{L}^{t\,*}(A)\| \leq \|A\|$, see (5.24). This dual map inherits the property to be a completely positive semigroup, which was established for Markovian dynamics (5.19), see [Dav1] and [Dav2]. By consequence, one gets for generator L^* the analogue of the Kossakowski-Lindblad representation [FrVe, AlFa]:

$$L^*(A) = i\,[H, A] + \frac{1}{2}\sum_{\alpha}(\widehat{V}_\alpha^*[A, \widehat{V}_\alpha] + [\widehat{V}_\alpha^*\,, A]\widehat{V}_\alpha)\ , \quad A \in \mathfrak{B}(\mathcal{H})\ , \qquad (5.29)$$

see (5.20), (5.21) and (5.28).

Duality (5.25) is useful for control the state evolution $\mathcal{L}^t(\rho)$ and, in particular, for the proof of the $t \to \infty$ limit ρ_∞ by calculation of this limit on observables. Since for any $\tau \geq 0$ and $\rho \in \mathfrak{C}_1(\mathcal{H})$, $A \in \mathfrak{B}(\mathcal{H})$:

$$\mathrm{Tr}_{\mathcal{H}}(\rho_\infty\, A) = \lim_{t\to\infty} \mathrm{Tr}_{\mathcal{H}}(\mathcal{L}^{\tau+t}(\rho), A) = \mathrm{Tr}_{\mathcal{H}}(\mathcal{L}^{\tau}(\rho_\infty), A)\ , \qquad (5.30)$$

we conclude that ρ_∞ is $\mathcal{L}^\tau$-invariant (*steady*) state. To elucidate topology of the density matrix convergence, recall that for the Kossakowski-Lindblad generator (5.20), (5.21) dynamics of state is trace-preserving (5.9): $\|\mathcal{L}^t(\rho)\|_{\mathfrak{C}_1} = 1$, and that (5.30) implies the *weak-operator* convergence $\mathcal{L}^t(\rho) \to \rho_\infty$. Then (5.30) is equivalent to the trace-norm convergence of density matrices:

$$\lim_{t\to\infty} \|\mathcal{L}^t(\rho) - \rho_\infty\|_{\mathfrak{C}_1} = 0\ , \qquad (5.31)$$

see e.g. [Za], Ch.2.4.

A.3 W^*-Dynamics, Steady States and Return to Equilibrium. For open cavity (5.5) the choice of operators $\{V_\alpha\}_\alpha$ in (5.29) is defined by (5.7). Since the cavity is a boson system, the natural (Fock) representation of the Canonical Commutation Relations (CCR) involves one-mode unbounded creation and annihilation operators b^* and b acting on the Hilbert space $\mathcal{H} = \mathcal{H}_C$. It is a boson Fock space $\mathcal{H} = \mathfrak{F}_B(\mathbb{C})$ over the one-dimensional subspace $\mathfrak{h} = \{\zeta\,\phi\}_{\zeta\in\mathbb{C}}$ (of a Hilbert space) corresponding to this one photon mode ϕ.

To avoid the problems with unbounded operators and to keep the evolution of observables in the space $\mathfrak{B}(\mathcal{H})$ one considers the corresponding $*$-algebra of bounded Weyl operators [BrRo2] in the form:

$$\mathfrak{W}(\mathbb{C}) := \left\{ W(\zeta) = \exp\left[\frac{i}{\sqrt{2}}\,(\overline{\zeta}\,b + \zeta\,b^*)\right]\right\}_{\zeta\in\mathbb{C}}\ , \qquad (5.32)$$

that verify the Weyl CCR-relations

$$W(\zeta_1)W(\zeta_2) = e^{-i\,\mathrm{Im}(\overline{\zeta_1}\,\zeta_2)/2}\,W(\zeta_1 + \zeta_2)\ , \qquad (5.33)$$

Note that the family $\{W(\zeta)\}_{\zeta\in\mathbb{C}}$ is continuous on $\mathcal{H}$ in the strong operator sense, but it is not continuous in the C^*-algebra topology: $\|W(\zeta) - \mathbb{1}\| = 2$ for any $\zeta \neq 0$.

By virtue of (5.7) and (5.21) one gets for the adjoint generator (5.29) of the one-mode open cavity with $H_{\mathcal{C}} = \epsilon\, b^* b$:

$$L^*_{\mathcal{C},\sigma}(A) = i\,[\epsilon\, b^* b, A] \tag{5.34}$$
$$+ \frac{1}{2}\left(\sigma_- b^*[A,b] + \sigma_-[b^*\,,A]b + \sigma_+ b[A,b^*] + \sigma_+[b\,,A]b^*\right)\ ,\quad A \in \mathfrak{B}(\mathcal{H})\ .$$

Then the adjoint semigroup equation:

$$\partial_t \mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(A) = \mathcal{L}^{t\,*}_{\mathcal{C},\sigma}\left(L^*_{\mathcal{C},\sigma}(A)\right)\ , \tag{5.35}$$

allows to calculate evolution of the Weyl operators (5.32): $A = W(\zeta)$, explicitly:

$$\mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(W(\zeta)) = e^{-\Omega_t(\zeta)}\ W(\zeta(t))\ . \tag{5.36}$$

Here

$$\Omega_t(\zeta) := \frac{|\zeta|^2}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\left\{1 - e^{-(\sigma_- - \sigma_+)t}\right\}\ ,\ \zeta(t) := \zeta\ e^{i\,\epsilon t - (\sigma_- - \sigma_+)t/2}\ . \tag{5.37}$$

Since $\|W(\zeta) - \mathbb{1}\| = 2$, the evolution (5.36) is not continuous in the C^*-algebra topology, but it does in the weak*-topology on the von Neumann algebra $\overline{\mathfrak{W}(\mathbb{C})}$ generated by (5.32) and the weak operator closure. Hence, the pair $(\overline{\mathfrak{W}(\mathbb{C})}, \mathcal{L}^{t\,*}_{\mathcal{C},\sigma})$ is W^*-dynamical system, see [BrRo1, AJPI].

Note that by differentiating of $\mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(W(\zeta))$ with respect to ζ and $\overline{\zeta}$ one can calculate the evolution of polynomials of creation-annihilation operators in the weak*-topology. For example of the photon number operator $N(t) := \mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(b^* b)$, the formal evolution equation follows directly from (5.34) for $A = b^* b$, and from (5.35):

$$\partial_t N(t) = -(\sigma_- - \sigma_+)N(t) + \sigma_+\ ,\ N(t=0) = b^* b\ . \tag{5.38}$$

If $\sigma_- > \sigma_+$ (leaking is stronger then pumping), then one gets

$$N(t) = e^{-(\sigma_- - \sigma_+)t}\ b^* b + \frac{\sigma_+}{\sigma_- - \sigma_+}\left\{1 - e^{-(\sigma_- - \sigma_+)t}\right\}\ . \tag{5.39}$$

By consequence, (5.39) formally implies

$$\lim_{t\to\infty} \mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(b^* b) = \mathbb{1}\ \frac{\sigma_+}{\sigma_- - \sigma_+}\ , \tag{5.40}$$

and (5.36) and (5.37) yield

$$w^* - \lim_{t\to\infty} \mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(W(\zeta)) = \mathbb{1}\ \exp\{-\frac{|\zeta|^2}{4}\ \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\}\ , \tag{5.41}$$

in the weak*-topology.

Let the initial cavity state ρ be such that $\overline{N}(0) := \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}}(\rho\, b^* b) < \infty$, and $\sigma_- > \sigma_+ > 0$. Then the limit (5.40) implies a nontrivial stationary expectation value of the photon number:

$$\overline{N}(t) := \lim_{t\to\infty} \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}}(\rho\, \mathcal{L}^{t\,*}(b^* b)) = \lim_{t\to\infty} \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}}(\mathcal{L}^{t}(\rho)\, b^* b) \tag{5.42}$$
$$= \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}}(\rho_\infty\, b^* b) = \frac{\sigma_+}{\sigma_- - \sigma_+}\ ,$$

in the limiting cavity state $\rho_\infty := \lim_{t\to\infty} \mathcal{L}^t(\rho)$. Similarly, by (5.41) one gets:

$$\lim_{t\to\infty} \mathrm{Tr}_{\mathcal{H}_\mathcal{C}}(\rho\,\mathcal{L}^{t\,*}(W(\zeta)) = \lim_{t\to\infty} \mathrm{Tr}_{\mathcal{H}_\mathcal{C}}(\mathcal{L}^t(\rho)\,W(\zeta)) \qquad (5.43)$$
$$= \mathrm{Tr}_{\mathcal{H}_\mathcal{C}}(\rho_\infty\,W(\zeta)) = \exp\{-\frac{|\zeta|^2}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\}\ .$$

By (5.32) and (5.42), (5.43) we obtain that the limiting (*steady*) density matrix ρ_∞ of the open cavity corresponds to the one-mode boson equilibrium state:

$$\rho_\infty =: \rho^{\beta_{cav}} = (1 - e^{-\beta_{\mathrm{cav}}\epsilon})\ e^{-\beta_{\mathrm{cav}}\epsilon\ b^*b}\ , \qquad \beta_{\mathrm{cav}} := \frac{1}{\epsilon}\ln\frac{\sigma_-}{\sigma_+}\ . \qquad (5.44)$$

Therefore, if $\sigma_- > \sigma_+ > 0$ and $\epsilon > 0$, the one-mode pumped leaky cavity evolves from *any* initial state verifying $\overline{N}(0) < \infty$, to the *equilibrium* Gibbs state with temperature $\theta_{cav} = 1/\beta_{cav}$ entirely defined by the leaking-pumping intensities $\sigma_\mp$. For this limit of return to the thermal equilibrium one can distinguish two intuitively clear extreme cases. The zero-temperature case for zero pumping: $\sigma_+ = 0$, and the *infinite* temperature case, when the pumping is not dominated by the leaking: $\sigma_+ \uparrow \sigma_-$. In the first case the photon-number mean-value (5.42) is zero, whereas it is infinite in the second case.

Note that the *time* and the *pumping* limits do not commute. Let the initial expectation of photons in the cavity $\overline{N}(0) < \infty$. Then applying the limit $\sigma_+ \uparrow \sigma_-$ to the expectation of (5.39) we obtain

$$\lim_{\sigma_+\uparrow\sigma_-} \overline{N}(t) = \overline{N}(0) + \sigma_+\ t\ . \qquad (5.45)$$

Therefore, one has a *linear* asymptotic increasing of the photon-number mean-value in the limit, when the leaking and pumping rates coincide: $\sigma_- = \sigma_+ > 0$.

A.4 Completely Positive Quasi-Free Dynamics on the CCR **Algebra.** We need a more abstract description of dynamics $\mathcal{L}^{t\,*}_{\mathcal{C},\sigma}$ on the CCR algebra $\overline{\mathfrak{W}(\mathbb{C})}$ generated by (5.32).

Notice that the complete positivity of the map $\mathcal{L}^{t\,*}_{\mathcal{C},\sigma} : \overline{\mathfrak{W}(\mathbb{C})} \to \overline{\mathfrak{W}(\mathbb{C})}$ follows from general properties of the Kossakowski-Lindblad generator (5.29). For the case of the open cavity this result follows directly from CCR-relations (5.33) and the explicit result (5.36). Behind this result there are abstract observations due to [DVV1, DVV2] and [EvLe2, Van].

Let $\mathfrak{H}$ be a Hilbert space. Denote by $\Gamma : \mathfrak{H} \to \mathfrak{H}$ a linear map and by $\Psi : \mathfrak{H} \to \mathbb{C}$ a complex function. Consider the Weyl CCR($\mathfrak{H}$) algebra generated by unitaries:

$$\mathfrak{W}(\mathfrak{H}) := \left\{ W(f) = \exp\left[\frac{i}{\sqrt{2}}\ (b(f) + b^*(f))\right]\right\}_{f\in\mathfrak{H}}\ . \qquad (5.46)$$

Here linear and anti-linear functions: $f \mapsto b^*(f)$ and $f \mapsto b(f)$, are creation and annihilation operators in the boson Fock space $\mathfrak{F}_B(\mathfrak{H})$ over $\mathfrak{H}$. Then the

Weyl CCR-relations take the form

$$W(f)W(g) = e^{-i\,\mathrm{Im}(f,g)_{\mathfrak{H}}/2}\, W(f+g)\ ,\ f,g \in \mathfrak{H}\ , \tag{5.47}$$

where $(\cdot,\cdot)_{\mathfrak{H}}$ is the scalar product in $\mathfrak{H}$.

Recall now that a linear map $T : \mathfrak{W}(\mathfrak{H}) \to \mathfrak{W}(\mathfrak{H})$ on the CCR($\mathfrak{H}$) algebra (5.46) is called *quasi-free*, if it has the form, [DVV1, DVV2]:

$$T(W(f)) = \Psi(f)\ W(\Gamma(f))\ ,\ f \in \mathfrak{H}\ . \tag{5.48}$$

Then the unity-preserving *quasi-free semigroup* $\{T_t\}_{t\geq 0}$ on the CCR($\mathfrak{H}$) algebra $\mathfrak{W}(\mathfrak{H})$ is defined in a similar way:

$$T_t(W(f)) := \Psi_t(f)\ W(\Gamma_t(f))\ ,\ f \in \mathfrak{H}\ , \tag{5.49}$$

where $\Psi_{t=0}(f) = 1$, $\Gamma_{t=0} = \mathbb{1}$ and $\Psi_t(f=0) = 1$, $\Gamma_t(f=0) = 0$. Note that the semigroup property of $\{T_t\}_{t\geq 0}$ and (5.49) imply

$$\begin{aligned} T_{s+t}(W(f)) &= T_s(T_t(W(f))) \\ &= \Psi_s(\Gamma_t(f))\Psi_t(f)\ W(\Gamma_s(\Gamma_t(f))) = \Psi_{s+t}(f)W(\Gamma_{s+t}(f))\ . \end{aligned} \tag{5.50}$$

Then linear independence of the Weyl operators yields

$$\Gamma_{s+t}(f) = \Gamma_s(\Gamma_t(f)) \quad \text{and} \quad \Psi_{s+t}(f) = \Psi_s(\Gamma_t(f))\Psi_t(f)\ . \tag{5.51}$$

Hence, $\{\Gamma_t\}_{t\geq 0}$ is in turn a semigroup on $\mathfrak{H}$.

Let $t \mapsto \Psi_t(f)$ be continuous and for each $f \in \mathfrak{H}$ be differentiable at $t = +0$:

$$\Psi_0'(f) := \lim_{t\to+0} \partial_t \Psi_t(f)\ , \tag{5.52}$$

such that the function $t \mapsto \Psi_0'(\Gamma_t(f))$ be bounded. Since semigroup (5.49) is unity-preserving, then (5.51) and $W(f=0) = \mathbb{1}$ imply that for any $f \in \mathfrak{H}$

$$\Psi_t(f) = \exp\left\{\int_0^t d\tau \Psi_0'(\Gamma_\tau(f))\right\}\ , \tag{5.53}$$

where $\Psi_0'(0) = 0$ and $\Psi_0'(-f) = \overline{\Psi_0'(f)}$, see [DVV2].

Note that for the particular case: $\mathfrak{H} = \mathbb{C}$ and for identification of (5.32) with (5.46), we reproduce the results (5.36) and (5.37) in **A.3** for $T_t = \mathcal{L}_{\mathcal{C},\sigma}^{t\,*}$. To this end one has to use (5.51), (5.52) and to put

$$\Gamma_t(\zeta) := \zeta\ e^{i\,\epsilon t-(\sigma_- - \sigma_+)t/2} \quad \text{and} \quad \Psi_0'(\zeta) := -\frac{|\zeta|^2}{4}(\sigma_- + \sigma_+)\ . \tag{5.54}$$

Then (5.53) gives $\Psi_t(\zeta) = e^{-\Omega_t(\zeta)}$, where $\Omega_t(\zeta)$ is defined by (5.37).

A.5 Quasi-Free States on the CCR **Algebra.** Since the linear hull of the the Weyl operators is dense in CCR($\mathfrak{H}$), any state $\omega(\cdot)$ is uniquely determined by its values taken on $\{W(f)\}_{f\in\mathfrak{H}}$. Therefore a state ω is completely defined by its characteristic functional

$$\mathfrak{H} \ni f \mapsto \omega(W(f)).$$

A state ω is called *regular* if the function $a \mapsto \omega(W(af))$ is continuous for all $f \in \mathfrak{H}$. Characteristic functionals of regular states on CCR($\mathfrak{H}$) are characterized by Araki and Segal [AJPI, BrRo1] in the following theorem.

Theorem 5.2. *A map* $\mathfrak{H} \ni f \mapsto \omega(W(f)) \in \mathbb{C}$ *is the characteristic functional of a regular state* ω *on CCR(*$\mathfrak{H}$*) if and only if*

1. $\omega(W(0)) = 1$.
2. *The function* $a \mapsto \omega(W(af))$ *is continuous for all* $f \in \mathfrak{H}$.
3. *For any integer* $n \geq 2$, *all* $f_1, ...f_n \in \mathfrak{H}$ *and all* $z_1, ...z_n \in \mathbb{C}$ *one has*

$$\sum_{j,k=1}^{n} \omega(W(f_j - f_k))e^{-i\,\mathrm{Im}\,(f_j,f_k)/2}\overline{z_j}z_k \geq 0.$$

We remind that the state $\omega_{r,s}(\cdot)$ is called *quasi-free* [Ver], if

$$\omega_{r,s}(W(f)) = \exp\{i\,r(f) - \frac{1}{2}\,s(f,f)\} \quad , \quad f \in \mathfrak{H} \ . \tag{5.55}$$

Here r is a linear functional on $\mathfrak{H}$, whereas s is a non-negative (closable) sesquilinear form on $\mathfrak{H} \times \mathfrak{H}$, that verifies

$$\frac{1}{4}\,|\mathrm{Im}(f,g)_{\mathfrak{H}}|^2 \leq s(f,f)\,s(g,g) \quad , \quad f,g \in \mathfrak{H} \ , \tag{5.56}$$

to ensure the positivity of this state [Ver]. By the Araki-Segal theorem the quasi-free states are *regular* and *analytic*, verifying the equations:

$$r(f) = \omega_{r,s}(\Phi(f)) \quad \text{and} \quad s(f,f) = \omega_{r,s}(\Phi(f)^2) - \omega_{r,s}(\Phi(f))^2 \ , \tag{5.57}$$

where $\Phi(f) := (b(f) + b^*(f))/\sqrt{2}$, see, e.g. [Ver, BrRo2].

If the state (5.55) is *gauge-invariant*: $\omega_{r,s}(W(f)) = \omega_{r,s}(W(e^{i\,\varphi}\,f))$, then $r(\cdot) = 0$ and we denote this state by $\omega_s(\cdot) := \omega_{r=0,s}(\cdot)$. By virtue of (5.43) the limiting (steady) state (5.44) of the open cavity is gauge invariant and quasi-free with:

$$r(\zeta) = 0 \ , \ s(\zeta,\zeta) = \frac{|\zeta|^2}{2}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{C}}}(\rho^{\beta_{cav}}\,\Phi(\zeta)^2) \ , \ \zeta \in \mathfrak{H} = \mathbb{C} \ , \tag{5.58}$$

Moreover it is the *equilibrium*, i.e. (β_{cav})-KMS state with respect to the Hamiltonian dynamics τ^t generated by $H_{\mathcal{C}}$.

It is worth to note that *a priori* there is no evidence that the steady state of the open cavity (5.5) must be either quasi-free, or the Gibbs equilibrium state. For discussion of the theory of *non-equilibrium* quasi-free steady states we suggest a very complete account in [AJPI]-[AJPIII]. There one can find curious examples of non-equilibrium (*non*-KMS) quasi-free states, which allow a certain informal Gibbs description via *long-range* many-body interactions [AschPi].

Note that by definitions (5.49) and (5.55) the quasi-free dynamics maps the quasi-free states into the states:

$$\omega_{r,s}(T_t(W(f))) = \Psi_t(f)\;\omega_{r,s}(W(\Gamma_t(f))) = \Psi_t(f)\;\omega_{r_t,s_t}(W(f)) \ , \tag{5.59}$$

where $r_t(f) := r(\Gamma_t(f))$ and $s_t(f,f) := s(\Gamma_t(f),\Gamma_t(f))$. In general the states (5.59) are not quasi-free.

Let us consider the case of *Gaussian* quasi-free dynamics [Van], when the semigroup $\{\Gamma_t\}_{t\geq 0}$ defined on $\mathfrak{H}$ by

$$\Gamma_t := \exp\{i\,tH - \frac{1}{2}\,t\,(\Sigma_- - \Sigma_+)\} \tag{5.60}$$

here H is a self-adjoint operator and $\Sigma_\mp$ are bounded positive operators on $\mathfrak{H}$ such that $\Sigma_- \geq \Sigma_+ \geq 0$, and (5.52) is a bilinear form

$$\Psi_0'(f) := -\frac{1}{4}\,(f, Rf)\ ,\ \ f \in \mathfrak{H}\ , \tag{5.61}$$

defined by a positive bounded operator $R \geq \Sigma_-$. Then by virtue of (5.53), (5.55) and (5.59) dynamics T_t maps initial quasi-free state $\omega_{r,s}$ into quasi-free state $\omega_{r_t,\widetilde{s}_t}$ with

$$\widetilde{s}_t(f,f) := s(\Gamma_t(f),\Gamma_t(f)) + \frac{1}{2}\int_0^t d\tau\,(\Gamma_\tau(f), R\,\Gamma_\tau(f))\ . \tag{5.62}$$

In the particular case of Hamiltonian dynamics ($\Sigma_\mp = 0$ and $R = 0$) the quasi-free map (5.49) is the group of Bogoliubov automorphisms on $\mathfrak{W}(\mathfrak{H})$:

$$T_t(W(f)) = W(e^{itH}f)\ ,\quad f \in \mathfrak{H}\ . \tag{5.63}$$

Automorphism (5.63) is the simplest quasi-free dynamics, which corresponds to the unitary one-particle evolution generated by the Hamiltonian H.

For the example of the open cavity, when $\mathfrak{H} = \mathbb{C}$, we use (5.54) to establish that in this case one has to put $H = \epsilon$, $\Sigma_\mp = \sigma_\mp$, $R = \sigma_- + \sigma_+$ in (5.60) and (5.61). Then (5.62) yields

$$\widetilde{s}_t(\zeta,\zeta) = s(\Gamma_t(\zeta),\Gamma_t(\zeta)) + \frac{1}{2}\,|\zeta|^2\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\,(1 - e^{-(\sigma_- - \sigma_+)t})\ . \tag{5.64}$$

This means that dynamics generated by (5.34) is quasi-free: $\mathcal{L}^{t\,*}_{\mathcal{C},\sigma}\,(W(\zeta)) = T_t(W(\zeta))$, see (5.54), and that it preserves the quasi-free states.

Since $\lim_{t\to\infty}\Gamma_t(\zeta) = 0$, by (5.64) we obtain that

$$\lim_{t\to\infty}\omega_{r,s}(\mathcal{L}^{t\,*}_{\mathcal{C},\sigma}(W(\zeta))) = \exp\{-\frac{|\zeta|^2}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\}\ , \tag{5.65}$$

for any initial quasi-free state $\omega_{r,s}$.

In fact we established in (5.43) that for $t \to \infty$ the quasi-free dynamics $\mathcal{L}^{t\,*}_{\mathcal{C},\sigma}$ transforms *any* initial state ρ with finite expectation of photon number into the *limit state*, which is quasi-free and Gibbs, see (5.43), (5.44) and (5.65).

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